A family of Coordinate Systems for Reissner-Nordstron Geometry

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Abstract

In this work we integrate Einstein-Maxwell equations for a point charge, in a coordinate system first considered by Lake. The family of solutions obtained includes various well known metrics. A free metric function acting as a gauge in the solution, is related to the Lorentz contraction factor in infinity, considering a set of geodesic frames.

It is well known that in the deduction of the metric of a static and spherically symmetric spacetime, in one of the intermediate steps, the following expression, is obtained,

$$ds^2 = -a(r)\,dt^2 + 2b(r)\,dr\,dt\, + c(r)\,dr^2 + r^2\,d\Omega^2, \tag{1}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\,d\phi^2$, is the metric of the 2-sphere, and $x^a = (t, r, \theta, \phi)$. At this point the freedom to select the time coordinate is usually invoked in order to eliminate the crossed term and end up with the standard relation in curvature coordinates; see for example [1].

Following Lake [2], we shall start directly from the line element (1) and integrate the Einstein-Maxwell equations corresponding to a charged point particle which, in the usual notation, are

$$
\begin{aligned}
G^a{}_b &= 2(F^a{}_c F^c{}_b + \frac{1}{4} F_{mn} F^{mn}) && \text{(2a)}\\
\nabla_b F^{ab} &= 0 && \text{(2b)}\\
\nabla_{[a} F_{bc]} &= 0, && \text{(2c)}
\end{aligned}
$$

where $\mathbf{G}$ is Einstein tensor and $\mathbf{F}$ is the electromagnetic field. By spherical symmetry $F^{01} = -F^{10}$ are the only non zero components of $\mathbf{F}$, so that (2c) is identically satisfied. On the other hand, (2b) can be written as

$$
\frac{1}{\sqrt{-g}} \partial_b (\sqrt{-g} F^{ab}) = 0, \tag{3}
$$

where $g = (ac + b^2)\, r^4 \sin^2\theta$ is the determinant of the metric defined by (1). After integration of (3) we obtain

$$
F^{01} = \frac{1}{\sqrt{ac + b^2}} \frac{q}{r^2}. \tag{4}
$$

We have identified the integration constant with the electric charge q. With this relation for F^{01} we can write the Einstein equations, which turn to be (2a),

$$
\begin{aligned}
\frac{ac - a + b^2 - a' r}{r^2 (ac + b^2)} &= -\frac{1}{2} \frac{q^2}{r^4} && \text{(5)}\\
\frac{b(ac' + ca' + 2bb')}{r(ac + b^2)^2} &= 0 && \text{(6)}\\
\frac{(ac + b^2)^2 - a^2\, c + r\, a^2\, c' - ab^2 + 2rabb' - rb^2 a'}{r^2 (ac + b^2)^2} &= \frac{1}{2} \frac{q^2}{r^4}. && \text{(7)}
\end{aligned}
$$

In these equations a prime denotes derivatives with respect to the radial coordinate r. Equation (6) can be integrated to obtain

$$
ac + b^2 = 1, \tag{8}
$$

where the integration constant has been chosen to insure that spacetime be asymptotically minkowskian. Using (8) in (5) the integration is easily performed, obtaining

$$
a(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}. \tag{9}
$$

The integration constant is $-2m$. Finally, we find the value of $b(r)$ from (8),

$$b(r) = \pm \left(1 - c(r) + \frac{2m}{r}c(r) - \frac{q^2}{r^2}c(r)\right)^{\frac{1}{2}}. \tag{10}$$

It can be verified that (7) is satisfied for these values of a and b, independently of the value of $c(r)$. In summary, the Einstein-Maxwell field equations admit as a solution the family of metrics, having $c(r)$ as a gauge function, given by the line element

$$ds^2 = -(1 - \frac{2m}{r} + \frac{q^2}{r^2})\, dt^2 \pm 2 \left(1 - c + \frac{2m}{r}c - \frac{q^2}{r^2}c\right)^{\frac{1}{2}} dr\, dt + c\, dr^2 + r^2 d\Omega^2. \tag{11}$$

It is important to stress that the choice of the gauge $c(r)$ amounts to the election of an observer and the corresponding temporal coordinate. For example, choosing $c = (1 - \frac{2m}{r} + \frac{q^2}{r^2})^{-1}$ and denoting the temporal coordinate by T, leads us to the Reissner-Nordström solution in the usual Schwarzschild standard coordinates

$$ds^2 = -f(r)\, dT^2 \ + f(r)^{-1}\, dr^2 + r^2 d\Omega^2 \tag{12}$$

with

$$f(r) = (1 - \frac{2m}{r} + \frac{q^2}{r^2}). \tag{13}$$

If instead we choose $c = 1 - \frac{2m}{r} + \frac{q^2}{r^2}$, the result is

$$ds^2 = -(1 - \frac{2m}{r} + \frac{q^2}{r^2})\, d\,\hat{t}^{\,2} + 2(\frac{2m}{r} - \frac{q^2}{r^2})\, dr\, d\,\hat{t} + (1 - \frac{2m}{r} + \frac{q^2}{r^2})\, dr^2 + r^2 d\Omega^2 \tag{14}$$

which is the expression reported by Papapetrou [3]. Selecting $c = 0$, and denoting the temporal coordinate by u, equation (11) becomes

$$ds^2 = -(1 - \frac{2m}{r} + \frac{q^2}{r^2})\, du^2 \pm 2dr\, du + r^2 d\Omega^2 \tag{15}$$

which is the Reissner-Nordström spacetime in the Eddington-Finkelstein null coordinate system [4].

Following [5] we shall suppose that c is a constant parameter restricted by $0 < c \leq 1$. We shall show that c is related with Lorentz contraction factor at infinity, and that the family of solutions moves continually between Painlevé-Gullstrand metric [6], [7] when ($c = 1$) and that of Eddington-Finkelstein ($c \to 0$) for Reissner-Nordström geometry. Let us consider the geometry of the spacetime described by the line element (12) and a free falling coordinate system from spacial infinity, radially towards the origin $r = 0$. This family of radial geodesics $[r(\tau), T(\tau), \theta = \text{constant}, \phi = \text{constant}]$ is determined by two conditions: first, the normalization of the 4-velocity of the geodesic observer, $\mathbf{u}.\mathbf{u} = -1$, and second, if $\mathbf{k}$ is a Killing vector, then $\mathbf{k}.\mathbf{u} = \text{constant}$ along the geodesic [8].

The first condition results, from (12),

$$f\ \dot{T}^2 - f^{-1}\,\dot{r}^2 = 1, \tag{16}$$

while for the time like Killing vector associated with symmetry under time translations, $k^a = \delta^a_0$, the second condition establishes that

$$f\quad \dot{T} = A \tag{17}$$

In these equation a dot denotes derivatives with respect to the proper time. In order to appreciate the meaning of A, let us express it in terms of the ordinary radial velocity $V \equiv \frac{\dot{r}}{\dot{T}}$. Using (16) and (17) the result can be written as follows:

$$A = \frac{f^{\frac{3}{2}}}{\sqrt{f^2 - V^2}}\ . \tag{18}$$

Evaluating this equation in $r = \infty$, where $f = 1$, we have

$$A = \frac{1}{\sqrt{1 - V^2}}, \tag{19}$$

which shows clearly that A is the Lorentz contraction factor in infinity. Let us solve (16) and (17) for the components $(\dot{t}, \dot{r})$ of $\mathbf{u}$. After lowering indices the result is

$$u_a = -A\delta^0_a - f^{-1}\,(A^2 - f)^{\frac{1}{2}}\,\delta^1_a, \tag{20}$$

thus u_a satisfy that $\partial_b u_a - \partial_a u_b = 0$, from which we get $u_a = -\partial_a \tilde{T}$. It is clear that the hypersurface $\tilde{T} = \text{constant}$ is the spatial section of the geodesic observer with 4-velocity u_a, therefore $\tilde{T}$ is the time coordinate associated with the free falling observer. Using (20) we get

$$d\tilde{T} = A\ dT + f^{-1}\,(A^2 - f)^{\frac{1}{2}}\,dr. \tag{21}$$

Solving for dT and using (12), the result is given by

$$ds^2 = -(1-\frac{2m}{r}+\frac{q^2}{r^2})\frac{d\tilde{T}^2}{A^2} + 2\sqrt{1-\frac{1}{A^2}(1-\frac{2m}{r}-\frac{q^2}{r^2})}\, dr\, \frac{d\tilde{T}}{A} + \frac{1}{A^2}\, dr^2 + r^2 d\Omega^2 \tag{22}$$

which, after redefining $\tilde{T}/A \rightarrow t$, matches with solution (11) which was obtained through the integration of the field equations, if we identify c with A^{-2}

Some final comments are in order. Note that for $c = 1$ ($A = 1$, no Lorentz contraction factor, zero initial velocity for the free falling observer), the resulting line element is

$$ds^2 = -(1-\frac{2m}{r}+\frac{q^2}{r^2})\, dt^2 + 2\sqrt{\frac{2m}{r}-\frac{q^2}{r^2}}\, dr\, dt + \, dr^2 + r^2 d\Omega^2 \tag{23}$$

which is the Reissner-Nordström geometry representation in the Painlevé-Gullstrand coordinates. Observe also that the general solution, equation (11), is not singular at the horizons of the Reissner-Nordström spacetime, $r_{\pm} = m \pm (m^2 - q^2)^{\frac{1}{2}}$, except for the Schwarzschild-like coordinates.

It is interesting to note that the family of metrics of the 3-space orthogonal to $\mathbf{u}$ are given by

$$ds^2_{(3)} = c\, dr^2 + r^2\, d\Omega^2 \tag{24}$$

thus, for the observer who starts from infinity with zero velocity ($c = A = 1$) the space is euclidean, corresponding to zero contraction of the radial distances. As we consider larger velocities at infinity, the contraction of the radial distances gets larger as well as the non euclidean character of space. In the limit $c \rightarrow 0$ (infinite Lorentz contraction, null geodesic) the 3-space collapses to the 2-sphere, $ds^2_{(2)} = r^2\, d\Omega^2$.

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