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On the definition of cylindrical symmetry

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Abstract. The standard definition of cylindrical symmetry in general relativity is reviewed. Taking the view that axial symmetry is an essential prerequisite for cylindrical symmetry, it is argued that the requirement of orthogonal transitivity of the isometry group should be dropped, this leading to a new, more general definition of cylindrical symmetry. Stationarity and staticity in cylindrically symmetric spacetimes are then defined, and these issues are analysed in connection with orthogonal transitivity, thus proving some new results on the structure of the isometry group for this class of spacetimes.

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1. Introduction

The purpose of this paper is to discuss the standard definition of cylindrically symmetric spacetimes and give some remarks on its possible generalizations. In particular, the assumptions which are usually made but are not necessary are pointed out, and the results presented herein will also be valid in some more general situations. Special attention is devoted to the *stationary and static* cylindrically symmetric cases.

The intuitive idea about cylindrical symmetry is very clear. However, there are some subtleties which deserve attention in general relativity. Just as an example we can remember that there are cases in which the axis of symmetry is spatially closed (a closed Robertson–Walker (RW) geometry, for instance), which may not seem to be in accordance with the standard view of a 'cylinder'. Our main assumption is that there is an axial Killing vector and that at least part of its axis of symmetry belongs to the spacetime. This will be absolutely essential for all our results. Of course, we could also consider situations where the axis of symmetry is completely absent, such as, for instance, when treating the exterior field for a cylindrical source. The axis is inside the source and thus the exterior field could be just any spacetime with a Killing vector having closed orbits. These Killing vectors can be obtained by identifying points in spacetimes with a spacelike symmetry, see also [1]. Nevertheless, our assumption is justified because any globally defined cylindrically symmetric spacetime will usually contain the axis.

Keeping the above assumption in mind, we need another spacelike symmetry such that the orbits of the G_2 group are *locally* cylinders, which must be assumed to be spacelike. The existence of 2-surfaces orthogonal to the group orbits is an extra assumption, not necessary

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for the definition of cylindrical symmetry, as we will see in a well known example, although in certain situations it holds as a consequence of the form of the Ricci tensor and the existence of the axis of symmetry. In summary, the basic ingredient for the cylindrical symmetry is a G_2 on S_2 group of motions containing an axial symmetry with the axis present in the given spacetime.

2. Axial and cylindrical symmetry

The purpose of this section is to review the definition of axial symmetry along with its associated basic geometrical features, and to put forward and discuss a definition of cylindrical symmetry, exploring its consequences.

Regarding axial symmetry, one has the following definition (see [2, 3]).

Definition 1. *A spacetime $(\mathcal{V}, g)$ is said to have axial symmetry if and only if there is an effective realization of the one-dimensional torus T into $\mathcal{V}$ that is an isometry and such that its set of fixed points is non-empty.*

Notice that definition 1 implicitly assumes that there exists at least one fixed point (i.e. points that remain invariant under the action of the group) in $(\mathcal{V}, \mathrm{g})$. In fact, it can be proven that the set of fixed points must be an autoparallel, two-dimensional timelike surface. This surface is the axis of symmetry and will henceforth be denoted as W_2 [2–5]. In previous standard definitions the axis was assumed to be a two-dimensional surface [4, 5], but, as we have just mentioned, this is necessarily so and therefore needs not be assumed as an extra requirement in the definition of axial symmetry [2, 3].

Furthermore, it can be shown [2, 3] that the infinitesimal generator $\vec{\xi}$ of the axial symmetry is spacelike in a neighbourhood of the axis, and that the so-called elementary flatness condition holds [2, 6], that is

$$\left.\frac{\nabla_\rho(\xi_\alpha\xi^\alpha)\nabla^\rho(\xi_\beta\xi^\beta)}{4\xi_\rho\xi^\rho}\right|_{W_2} \longrightarrow 1. \qquad (1)$$

This condition ensures the standard 2π-periodicity of the axial coordinate near the axis.

Further fundamental results concern the relation of the Killing vector $\vec{\xi}$ with other vector fields, and in particular with different isometry generators. We refer to [2–5] for proofs.

Theorem 1. *Let $\vec{v}$ be a vector field in an axisymmetric spacetime and $q \in W_2$.*

(a) $\vec{v}|_q$ is tangent to the axis at q iff $[\vec{v}, \vec{\xi}]|_q = 0$.
(b) $\vec{v}|_q$ $(\neq 0)$ is normal to the axis at q iff $\vec{v}|_q$ and $[\vec{v}, \vec{\xi}]|_q$ are linearly independent vectors and $[[\vec{v}, \vec{\xi}], \vec{\xi}]|_q$ depends linearly on the previous vectors.
(c) $\vec{v}$ is neither tangent nor normal to the axis at q iff $\vec{v}|_q$, $[\vec{v}, \vec{\xi}]|_q$ and $[[\vec{v}, \vec{\xi}], \vec{\xi}]|_q$ are linearly independent vectors and $[[[\vec{v}, \vec{\xi}], \vec{\xi}], \vec{\xi}]|_q$ depends linearly on the previous two. ■

Theorem 2. *In an axially symmetric spacetime, if $\vec{\lambda}$ is a Killing vector field tangent to the axis of symmetry for all $q \in W_2$, then*

$$[\vec{\xi}, \vec{\lambda}] = 0.$$ ■

Proposition 1. *In an axisymmetric spacetime, let $\vec{\lambda}$ be a Killing vector field which does not commute with $\vec{\xi}$. If at some point q of the axis $\vec{\lambda}|_q$ is not normal to W_2, then there always exists another Killing vector field given by $\vec{\lambda} + [[\vec{\lambda}, \vec{\xi}], \vec{\xi}]$ that commutes with $\vec{\xi}$, and is therefore tangential to the axis.* ■

It should be noted that all of the above results also apply to conformal Killing vector fields [2, 3].

Let us next consider the definition of cylindrical symmetry. In addition to the existence of two spacelike Killing vector fields, $\vec{\xi}$ and $\vec{\eta}$, one of which, say $\vec{\xi}$, is taken to generate an axial symmetry, it has usually been assumed that both Killing vectors commute and that the G_2 acts orthogonally transitively. With regard to the assumption of commutativity, from proposition 1 it is clear that the existence of a Killing vector field that is not orthogonal to W_2 at some point would suffice. However, not even this assumption is actually necessary due to the following result.

Proposition 2. *In an axially symmetric spacetime, if there is another Killing vector $\vec{\lambda}$ which generates with $\vec{\xi}$ a G_2 group, then both Killing vectors commute, thus generating an Abelian G_2 group.*

Proof. If $\vec{\lambda}|_q$ is not orthogonal to the axis for a given point $q \in W_2$, then from proposition 1 we have that the vector field $\vec{\lambda} + [[\vec{\lambda}, \vec{\xi}], \vec{\xi}]$, which belongs to the same G_2 group, commutes with $\vec{\xi}$, leading to an Abelian G_2 group. Suppose then that $\vec{\lambda}$ is orthogonal to W_2 at all its points. From theorem 1 point (b) it follows that another independent Killing vector field given by $\vec{\lambda}' \equiv [\vec{\xi}, \vec{\lambda}]$ exists; but this leads to a contradiction because we are under the assumption that $\vec{\xi}$ and $\vec{\lambda}$ generate a group of isometries. □

The assumption on the existence of 2-surfaces orthogonal to the group orbits (see, for instance, [1, 6–8]) is not necessary for the definition of cylindrical symmetry or a consequence of it, as we will see in an explicit example below, although its justification would come mainly from three different sorts of reasons. The first one concerns the invertibility of the G_2 group, which is equivalent to its orthogonal transitivity [9]. The second corresponds to the considerations given by Melvin [7, 8] about the invariance under reflection in planes containing the axis and perpendicular to it (this is used explicitly in the definition of the whole cylindrical symmetry, that is, such that $\vec{\xi}$ and $\vec{\eta}$ are also mutually orthogonal). This is, in fact, equivalent to demanding the invertibility of each of the one-parameter subgroups forming the Abelian G_2, and thus it is a particular case of the first assumption. The previous reasoning is geometrical in nature, while the third is based on results concerning the form of the Ricci tensor for some interesting material contents, such as Λ-terms (including vacuum) and perfect fluids whose velocity vector $\vec{u}$ is orthogonal to the group orbits, since in those cases it can be shown (see [5, 6, 9]), on account of the vanishing of $\vec{\xi}$ at W_2, that orthogonal transitivity follows.

The possible definition for cylindrically symmetric spacetimes, avoiding complementary assumptions, could thus be:

Definition 2. *A spacetime $(\mathcal{V}, g)$ is cylindrically symmetric if and only if it admits a G_2 on S_2 group of isometries containing an axial symmetry.*

The line-element of cylindrically symmetric spacetimes corresponds then to that of the Abelian G_2 on S_2 spacetimes [10], since this definition automatically implies that the G_2 group must be Abelian as follows from proposition 2 above. Orthogonal transitivity is then left as an extra assumption, taking into account that, as was already mentioned, it follows directly in some important cases from the structure of the Ricci tensor and the existence of an axis.

The above definition is inspired by the intuitive idea of cylindrically symmetric spacetimes as those containing spatial cylinders, which are just $S^1 \times V_1$ spacelike surfaces *with a flat metric*. Here, by V_1 we mean any of S^1 or $\mathbb{R}$ spaces, that is, we consider not only a spatially infinite axis of symmetry, but also a spatially finite axis which may appear (as in a closed RW geometry).

Notice that these $S^1 \times S^1$ surfaces are *not* standard toruses since the first fundamental form of a standard torus is non-flat.

Non-orthogonally transitive Abelian G_2 on S_2 spacetimes with an axial symmetry (metrics of types A(i) and A(ii) in Wainwright's classification [10]) must be considered as cylindrically symmetric as they contain a two-parameter family of embedded spatial cylinders. A well known explicit example is given by the dust spacetime with line-element (equation (20.13) in [6])

$$ds^2 = e^{-a^2\rho^2}\left(d\rho^2 + dz^2\right) + \rho^2\, d\varphi^2 - \left(dt + a\rho^2\, d\varphi\right)^2, \tag{2}$$

which belongs to the van Stockum class of stationary axisymmetric dust spacetimes [11], whose G_2 on S_2 group is non-orthogonally transitive. The fluid flow (tangential to $\partial/\partial t$) is not orthogonal to the group orbits, as otherwise the orthogonal transitivity would follow necessarily from the perfect-fluid form of the Ricci tensor and the existence of the axis of symmetry (see above). The spacelike character of the axial Killing vector is ensured in a region around the axis. The spacetime can then be matched to a static vacuum metric [6]. The surfaces given by $\{t = \text{constant}, \rho = \text{constant}\}$ constitute the two-parameter family of embedded spatial cylinders, which are rigidly rotating around the axis of symmetry ($\rho = 0$).

This situation regarding the orthogonal transitivity in cylindrical symmetry is clearly in contrast with the case of spherical symmetry, where the existence of surfaces orthogonal to the group orbits is ensured *geometrically* [6, 12].

3. Stationary and static cylindrically symmetric spacetimes

Once we have discussed cylindrical symmetry, we can proceed further and study the definitions of both stationary and static cylindrically symmetric spacetimes. Stationarity implies the existence of an additional isometry which is generated by a timelike Killing vector field (that is integrable in the static case). The first consequence is that, since a timelike vector field cannot be orthogonal to W_2 anywhere, proposition 1 and theorem 2 imply the existence of a Killing vector $\vec{\zeta}$ such that $[\vec{\xi}, \vec{\zeta}] = 0$ which can be checked to be timelike in the region where the original one was timelike [2, 4, 5]. Therefore, at this stage, we have that the group structure of stationary cylindrically symmetric spacetimes is a G_3 on T_3 group of isometries generated by two spacelike Killing vectors $\vec{\xi}$ and $\vec{\eta}$, and a timelike Killing vector $\vec{\zeta}$†, such that $\vec{\xi}$ commutes with both $\vec{\eta}$ and $\vec{\zeta}$,

$$[\vec{\xi}, \vec{\eta}] = 0, \qquad [\vec{\xi}, \vec{\zeta}] = 0. \tag{3}$$

In the static case we must further impose the existence of an integrable timelike Killing vector $\vec{s}$. It should be noticed that in the static case, $\vec{s}$ does not necessarily coincide with $\vec{\zeta}$ in principle.

Notice that the definition of stationary cylindrically symmetric spacetimes which appears in [6] includes the extra assumption $[\vec{\eta}, \vec{\zeta}] = 0$, apart from the orthogonal transitivity on the G_2 on S_2 assumed in the definition of cylindrical symmetry in this reference. However, as we will see later in the next section, the assumption $[\vec{\eta}, \vec{\zeta}] = 0$ together with orthogonal transitivity of the G_2 on S_2 subgroup implies, in fact, staticity. Therefore, in order to look for actual stationary (non-static) models of these characteristics, one of the two extra assumptions must be dropped.

As a matter of fact, in [6] the extra assumption $[\vec{\eta}, \vec{\zeta}] = 0$ is maintained instead of the orthogonal transitivity on the G_2 on S_2 subgroup, stating essentially that (the phrasing is ours)

† Although the existence of such a timelike Killing vector field in a spacetime with a G_3 on T_3 is not ensured globally, it is certainly true in some open neighbourhood of any given point.

'stationary cylindrically symmetric spacetimes are those admitting an Abelian G_3 on T_3 group of isometries containing a G_2 on S_2 subgroup with an axial symmetry'. Of course, this is not coherent with the assumption of orthogonal transitivity in the definition of cylindrical symmetry that appears in the same reference. Indeed, the metric (2) is presented in [6] as an example of stationary cylindrically symmetric spacetime, although its G_2 on S_2 group does not act orthogonally transitively. Let us remark that the metrics appearing in section 20.2 of [6], which are presented as stationary cylindrically symmetric *vacuum* solutions, also possess a G_2 on S_2 which is not orthogonally transitive, but this necessarily implies that the axial symmetry cannot be well defined in these *vacuum* spacetimes, as we have mentioned in the previous section. Nevertheless, all these vacuum examples could be matched to another cylindrically symmetric spacetime with the axis included, which would then be considered as the source of the exterior vacuum spacetime, so that the axis of symmetry would not be present in the vacuum region.

In the next section we will focus on the assumption that the G_2 on S_2 acts orthogonally transitively, which will give some results concerning the group structures and the form of the line-elements. This study also has a clear motivation, since in some relevant aforementioned cases (including vacuum) this assumption is a direct consequence.

4. Stationarity, staticity and orthogonal transitivity in cylindrically symmetric spacetimes

Let us now assume that $\vec{\xi}$ and $\vec{\eta}$ generate an Abelian subgroup G_2 whose orbits S_2 admit orthogonal surfaces, i.e.

$$\boldsymbol{\xi} \wedge \boldsymbol{\eta} \wedge \mathrm{d}\boldsymbol{\xi} = 0, \qquad \boldsymbol{\xi} \wedge \boldsymbol{\eta} \wedge \mathrm{d}\boldsymbol{\eta} = 0.$$

It is straightforward to show that there are four non-isomorphic algebraic structures for the G_3 group generated by $\{\vec{\xi}, \vec{\zeta}, \vec{\eta}\}$ satisfying (3) [2, 3, 13–16], and taking into account that $\vec{\xi}$ vanishes on the axis, the remaining commutator can then be expressed, in each case and without loss of generality, as

(a) Abelian case: (Bianchi I) $[\vec{\eta}, \vec{\zeta}] = 0$;
(b) case I: (Bianchi III) $[\vec{\eta}, \vec{\zeta}] = b\vec{\zeta}$;
(c) case II: (Bianchi III) $[\vec{\eta}, \vec{\zeta}] = c\vec{\eta}$;
(d) case III: (Bianchi II) $[\vec{\eta}, \vec{\zeta}] = d\vec{\xi}$,

where b, c and d are constants. Some of these constants could have been set equal to 1 by conveniently rescaling the Killing vectors, but we choose not to do so because they can carry physical units. Notice that the above algebraic structure does not depend on the timelike or spacelike character of the Killing vector field $\vec{\zeta}$. Now, taking into account that $\vec{\xi}$ and $\vec{\eta}$ span a subgroup which acts orthogonally transitively and using the fact that we want the G_3 group acting on T_3, so that the projection of the globally defined Killing vector field $\vec{\zeta}$ onto the surfaces *orthogonal* to the orbits generated by the G_2 subgroup $\{\vec{\xi}, \vec{\eta}\}$ is necessarily timelike, it follows that we can choose coordinates $\{t, x, \varphi, z\}$ such that

$$\vec{\xi} = \frac{\partial}{\partial \varphi}, \qquad \vec{\eta} = \frac{\partial}{\partial z}, \tag{4}$$

and the line-elements for each of the above algebras can then be written as follows (see [3, 13–16]).

Abelian case. The line-element is given by

$$\mathrm{d}s^2 = \frac{1}{S^2(x)}\left[-\mathrm{d}t^2 + \mathrm{d}x^2 + \frac{Q^2(x)}{F(x)}\,\mathrm{d}\varphi^2 + F(x)(\mathrm{d}z + W(x)\,\mathrm{d}\varphi)^2\right], \tag{5}$$

where S, Q, F and W are arbitrary functions of x and the Killing vector $\vec{\zeta}$ has the following expression:

$$\vec{\zeta} = \frac{\partial}{\partial t}.$$

In this case we can choose $\vec{\zeta} = \vec{s}$ because $\vec{\zeta}$ is already an integrable timelike Killing vector field and thus we have a static cylindrically symmetric spacetime. This indirectly proves the following:

Proposition 3. *Given an Abelian G_3 on T_3 containing a subgroup G_2 on S_2 acting orthogonally transitively, there always exists an integrable timelike Killing vector field.* ■

This applies, in fact, for G_2 on V_2 and an additional conformal Killing vector field with the 'opposite' character [3, 16]. Therefore, we have

Corollary 3.1. *A (non-static) stationary spacetime cannot contain an orthogonally transitive Abelian G_2 on S_2 subgroup whenever the G_3 group containing these symmetries is Abelian.*

Case I. The line-element now takes the form

$$\mathrm{d}s^2 = \frac{1}{S^2(x)}\left[-\mathrm{d}t^2 + \mathrm{d}x^2 + b^2M^2(t)\,\mathrm{d}z^2 + L^2(x)(\mathrm{d}\varphi + bN(t)\,\mathrm{d}z)^2\right],$$

where M and N are functions of t satisfying $M_{,t}^2 = 1 + \alpha M^2$ with $M_{,t} \neq 0$, $N_{,t} = \omega M$, α, ω are constants, and L is an arbitrary function of x. The Killing vector $\vec{\zeta}$ reads

$$\vec{\zeta} = \mathrm{e}^{bz}\left(-\frac{1}{b}\frac{M_{,t}}{M}\frac{\partial}{\partial z} + \left(N\frac{M_{,t}}{M} - \omega M\right)\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial t}\right). \tag{6}$$

This vector field is timelike if $\alpha + L^2\omega^2 < 0$.

Case II. In this case the line-element has the following expression:

$$\mathrm{d}s^2 = \frac{1}{S^2(x)}\left[-\mathrm{d}t^2 + \mathrm{d}x^2 + \frac{Q^2(x)}{F(x)}\,\mathrm{d}\varphi^2 + F(x)\left(\mathrm{e}^{-ct}\,\mathrm{d}z + W(x)\,\mathrm{d}\varphi\right)^2\right],$$

and we have then

$$\vec{\zeta} = cz\frac{\partial}{\partial z} + \frac{\partial}{\partial t},$$

which is timelike whenever $c^2z^2F\mathrm{e}^{-2ct} - 1 < 0$.

Case III. The line-element now reads

$$\mathrm{d}s^2 = \frac{1}{S^2(x)}\left[-\mathrm{d}t^2 + \mathrm{d}x^2 + F(x)\,\mathrm{d}z^2 + \frac{Q^2(x)}{F(x)}(\mathrm{d}\varphi + (W(x) - td)\,\mathrm{d}z)^2\right],$$

and $\vec{\zeta}$ is given by

$$\vec{\zeta} = zd\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial t},$$

being timelike in the region $z^2d^2Q^2 - F < 0$.

The only cases in which the timelike character of the Killing vector $\vec{\zeta}$ is ensured all over the spacetime are the Abelian case and also in case I, once the function $L(x)$ has been chosen appropriately.

In order to see whether or not a globally defined timelike Killing vector field exists in the non-Abelian cases, we consider a general Killing vector $\vec{s}$ not contained in the G_2, i.e.

$$\vec{s} = \vec{\zeta} + A\vec{\xi} + B\vec{\eta}, \tag{7}$$

where A and B are arbitrary constants, and compute its modulus in each case. It follows:

$$\begin{aligned}
&\text{case I:} && (\vec{s}\cdot\vec{s}) = \frac{1}{S^2}\big\{\mathrm{e}^{2bz}M^2(\alpha + L^2\omega^2) - 2M\mathrm{e}^{bz}\big[A\omega L^2 + Bb\big(\omega NL^2 + M_{,t}\big)\big]\\
& && \qquad + L^2(A + bBN)^2 + B^2b^2M^2\big\}\\
&\text{case II:} && (\vec{s}\cdot\vec{s}) = \frac{1}{S^2}\left\{-1 + A^2\frac{Q^2}{F} + F\big(cz\mathrm{e}^{ct} + AW\big)^2\right\},\\
&\text{case III:} && (\vec{s}\cdot\vec{s}) = \frac{1}{S^2}\left\{-1 + B^2F + \frac{Q^2}{F}(zd + B(W - td))^2\right\}.
\end{aligned} \tag{8}$$

From the above expressions it can immediately be seen that in cases II and III, and for any given functions of x and constants A and B, we can reach points where $(\vec{s}\cdot\vec{s}) > 0$ whenever the coordinate z can reach any value in $(-\infty, \infty)$. Therefore, stationary spacetimes with a globally defined timelike Killing vector field whose axis of symmetry extends indefinitely in the z-coordinate can only exist in the Abelian case or in some situations of case I.

Let us next investigate the existence of integrable Killing vectors in cases I–III. If one such vector field outside the G_2 group exists, $\vec{s}$, it must be of the form given by (7) although it will not be supposed to be timelike *a priori*. The 1-form $\boldsymbol{s}$ has the following form, common to all three cases: $S^2(x)\boldsymbol{s} = s_0(z)\,\mathrm{d}t + s_2(t,x,z)\,\mathrm{d}\varphi + s_3(t,x,z)\,\mathrm{d}z$ with $s_0 \neq 0$, so that the condition $\boldsymbol{s}\wedge\mathrm{d}\boldsymbol{s} = 0$ gives the following three equations:

$$s_{2,x} = s_{3,x} = 0, \qquad s_0s_{2,z} + s_2(s_{3,t} - s_{0,z}) - s_3s_{2,t} = 0. \tag{9}$$

Let us impose these conditions on each of the cases under study.

Case I. Equations (9) imply first $L_{,x}\omega = 0$. If we take $\omega \neq 0 \Rightarrow L = L_0$ (constant), but since the axis of symmetry W_2 is given by those points for which $L(x) = 0$, it follows that $L_0 = 0$ which is inconsistent with the dimension of the spacetime, therefore it must be $\omega = 0$.

As $\omega = 0 \Rightarrow N = N_0$ (constant), but in that case it is easy to see that the coordinate change $\varphi + bN_0z \mapsto \varphi$, which preserves the form of the axial Killing, renders the metric in diagonal form,

$$\mathrm{d}s^2 = \frac{1}{S^2(x)}\big[-\mathrm{d}t^2 + b^2M^2(t)\,\mathrm{d}z^2 + \mathrm{d}x^2 + L^2(x)\,\mathrm{d}\varphi^2\big],$$

which can be further transformed by suitably redefining the coordinate x to the form

$$\mathrm{d}s^2 = \frac{1}{S^2(x)}\left[-\mathrm{d}t^2 + b^2M^2(t)\,\mathrm{d}z^2\right] + \mathrm{d}x^2 + L^2(x)\,\mathrm{d}\varphi^2,$$

which is that of a (class B) warped spacetime (see [17]) and can be easily seen to admit a larger group of isometries: at least G_4 on T_3. In this case, equations (9) readily imply $A = 0$, and a calculation of the modulus of $\vec{s}$, gives

$$(\vec{s}\cdot\vec{s}) = \frac{1}{S^2}\left\{-\mathrm{e}^{2bz} + \left(M_{,t}\mathrm{e}^{bz} - MbB\right)^2\right\};$$

thus, for spacetimes whose range for the z coordinate is not bounded, we have $(\vec{s}\cdot\vec{s}) > 0$ when $bz \to -\infty$ unless we set $B = 0$, and therefore we have $\vec{s} = \vec{\zeta}$ which is timelike in the whole manifold iff $\alpha < 0$.

Therefore, the existence of a (timelike) integrable Killing vector implies, for this class of spacetimes, the existence of (at least) a G_4 on T_3 group of isometries which contains the original G_3 on T_3, as well as a subgroup G_3 acting on timelike two-dimensional orbits of constant curvature.

It is easy to see that the Segre type of the Ricci (or Einstein) tensor is $\{(1,1)11\}$ or some degeneracy thereof, whereas the Petrov type of the Weyl tensor is, in general, D.

Case II. A shift in the coordinate z allows us to put $B = 0$ without loss of generality. Equations (9) imply $A = W = F' = 0$, so that $\vec{s} = \vec{\zeta}$ and the metric can then be written as:

$$\mathrm{d}s^2 = \frac{1}{S^2(x)}\left[-\mathrm{d}t^2 + \exp\left(-2ct\right)\mathrm{d}z^2\right] + \mathrm{d}x^2 + Q^2(x)\,\mathrm{d}\varphi^2,$$

where the x coordinate has been redefined in an obvious way. It then follows that this is again a type-B warped spacetime which admits a group G_4 on T_3 of isometries which contains the G_3 on T_3, and also as in the previous case, a subgroup G_3 acting on timelike two-dimensional orbits of constant curvature.

As in case I, the Ricci tensor is of the Segre type $\{(1,1)11\}$ or some degeneracy thereof, and the Weyl tensor is type D.

Case III. Analogously to the previous case, we can put $A = 0$ without loss of generality. In this case, however, equations (9) imply $d = 0$, which leads to the Abelian case, thus no timelike integrable Killing vector exists in this group. As a matter of fact, what we have just proven is slightly more general than this; we summarize the results in the following.

Proposition 4. *Given a G_3 on T_3 group of Bianchi type II having an Abelian G_2 subgroup acting orthogonally transitively and containing an axial Killing vector, then the only integrable Killing vectors in this group belong to the subgroup G_2.* ∎

This result can also be obtained for a conformal group C_3 containing a G_2.

The definition of stationary (or static) cylindrically symmetric spacetimes has been based on that for cylindrically symmetric spacetimes. If, on the other hand, we had started with the usual definition of stationary axisymmetric spacetimes [6], we could have imposed that the timelike Killing vector $\vec{s}$ and the axial Killing vector $\vec{\xi}$ generate a G_2 on T_2 group acting orthogonally transitively (for non-convective rotating fluids [18, 19], see for instance [20]), and the allowed Lie algebra structures would then be those four discussed previously. Nevertheless, it can be shown that the imposition of orthogonal transitivity on the orbits generated by $\vec{s}$ and $\vec{\xi}$ gives no further restriction in the *static* non-Abelian cases. The conditions

$$\boldsymbol{\xi} \wedge \boldsymbol{s} \wedge \mathrm{d}\boldsymbol{\xi} = 0, \qquad \boldsymbol{\xi} \wedge \boldsymbol{s} \wedge \mathrm{d}\boldsymbol{s} = 0 \tag{10}$$

applied to each of the algebraic cases give

Abelian case: $2(F'Q - FQ')QW - (F^2W^2 - Q^2)W'F = B(F'Q^2 - W'F^3W) = 0$

case I: automatically satisfied

case II: $W'Q^2 + F^2W^2W' - 2WQQ' = F'Q^2 - W'F^3W = 0$

case III: $W' = BF' = 0.$

Clearly, the conditions for the existence of a timelike integrable Killing vector in case II (which turns out to be $\vec{\zeta}$) imply that the orbits generated by $\vec{s}$ and $\vec{\xi}$ admit two-dimensional orthogonal surfaces as well as the existence of (at least) a fourth linearly independent Killing vector which, along with the previous three, generates a group G_4 on orbits T_3.

Therefore, the assumption (10) gives no further restrictions in the non-Abelian cases, neither when imposing a timelike integrable Killing in a geometrical sense (i.e. including case II), nor in the stationary case (when cases II and III could be avoided).

All of the above can be summarized in the following.

Theorem 3. *Given a G_3 on T_3 group that contains an orthogonally transitive Abelian G_2 subgroup generated by an axial $\vec{\xi}$ and $\vec{\eta}$, then it follows:*

(a) If G_3 is the maximal isometry group, then it must be Abelian.
(b) If G_3 is non-Abelian, then it is (locally) contained in a G_4 on T_3, and $\vec{\xi}$ and $\vec{\zeta}$ generate an orthogonally transitive subgroup G_2 on T_2. ■

Notice that, in addition, in these non-Abelian cases there exist two-dimensional timelike surfaces of constant curvature, the Segre type of the Ricci tensor is $\{(1, 1)11\}$ or some degeneracy thereof, and the Petrov type is, in general, D.

5. Conclusions

Let us briefly summarize the main points and results of this paper. The definition of cylindrical symmetry as given in standard reference texts (see, e.g., [6] and section 2) has been analysed critically. Given that axial symmetry is an essential prerequisite, and *assuming* that (at least part of) the axis is present in the spacetime, we can use the proper definition of axial symmetry as appearing in [2–5]; that is, a one-parameter isometry group with closed orbits whose set of fixed points, or axis, is *non-empty*.

Under these hypotheses, we show (see proposition 2) that the assumption of commutativity of the two Killing vector fields of a cylindrically symmetric spacetime is superfluous, since it follows from the existence of a G_2 containing an axial symmetry. We next argue that orthogonal transitivity is neither necessary for, nor a consequence of, a sensible definition of cylindrical symmetry, even though it arises quite naturally in a wide range of situations of interest in relativity. We illustrate this point with an explicit example. As a result of the foregoing discussion, we put forward a new definition of cylindrical symmetry, see definition 2, as that implied by the mere existence of a G_2 on S_2 group of isometries containing an axial symmetry, without any further restrictions.

In section 3, we address the problem of stationarity and staticity in cylindrically symmetric spacetimes. Again, we review the definition given in [6], showing that it is not fully consistent with that of cylindrical symmetry in the same reference. Here the problem arises because in the original definition of [6] orthogonal transitivity was assumed, but then it is tacitly dropped without mention. Actually, in section 4 we prove that the addition of orthogonal transitivity implies, in fact, staticity, so that there *cannot* be any stationary (non-static) cylindrically symmetric spacetime with an orthogonally transitive G_2 on S_2.

More specifically, section 4 is devoted to the case of stationary and cylindrically symmetric spacetimes where the G_2 on S_2 acts orthogonally transitively, as happens in many situations of physical interest, see the remarks at the beginning of this section. We classify the possible three-dimensional Lie algebra structures into four different types, and show (see theorem 3) that under these assumptions, if the G_3 is the maximal isometry group of the spacetime, then it must be Abelian and the spacetime is, in fact, static; and also that if the G_3 is non-Abelian, then it is locally contained in a G_4 which acts on the same orbits T_3 as the G_3. These orbits contain in turn two-dimensional timelike surfaces of constant curvature, and the isotropy then imposes restrictions on the Petrov and Segre types.

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