Non-abelian self-duality from self-interaction

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Abstract

The non-abelian self-dual action in three dimensions is derived using the self-interaction mechanism.

Self-duality in three dimensions was proposed initially by Townsend et. al. [1] as an alternative to the topologically massive theory[2]. In principle, they seem different descriptions of a locally massive spin 1 physical excitation: the self-dual theory is described by a non-gauge invariant first order action while the topologically massive action is written down in a gauge invariant second order formulation. Both actions have an abelian Chern-Simons term ($\epsilon^{mnp}A_m\partial_n A_p$). Despite these differences, Deser and Jackiw stablished that both theories are locally equivalent through the existence of a master action, even in the presence of external sources[3]. Moreover, both theories are dual equivalent[4] and the self-dual theory can be seen as a gauged fixed version of the topologically massive theory[5]. The self-dual theory for gravity and for higher spin in three dimensions was achieved in [6] and [7], respectively. If glogal properties are considered, the equivalence is modified, for instance, the partition functions of the self dual and topologically massive theories are not the same but they are related in the following way: $Z_{SD} = Z_{CS}Z_{TM}$[8] (where Z_{CS} is the partition function of the abelian Chern-Simons action).

The non-abelian generalization of the topologically massive theory was given in [2] while the non-abelian self-dual theory was formulated independently by McKeon [9] and Arias, et. al.[10], which has a structure of a Freedman-Townsend action[11].

In this letter, starting from an appropiate master action, we will derive the non-abelian self-dual action using the self-interaction mechanism[12].

We will start by considering the following master action[13]

$$I = \int d^3x[-\mu\epsilon^{mnp}A_m\partial_n a_p - \frac{1}{2}\mu^2 a_m a^m - \mu\epsilon^{mnp}A_m\partial_n v_p + \frac{1}{2}\mu\epsilon^{mnp}v_m\partial_n v_p] \quad (1)$$

This action can be seen as the coupling between a Maxwell field (A_m) and a vector field (v_m) described by an abelian Chern-Simons action through a three dimensional BF topological term. Independent variations in the a_m, v_m and A_m fields, yield the following equations of motion

$$a^m = -\frac{1}{2}\mu\epsilon^{mnp}f_{np(A)}, \quad (2)$$

$$\epsilon^{mnp}\partial_n[A_p - v_p] = 0 \quad (3)$$

and

$$\epsilon^{mnp}\partial_n[a_p + v_p] = 0, \quad (4)$$

where $f_{mn(A)} = \partial_m A_n - \partial_n A_m$. The last two equations can be solved locally. We have

$$v_m = A_m + \partial_m\phi \quad (5)$$

and

$$a_m = -v_m + \partial_m\sigma. \quad (6)$$

The master action has abelian gauge invariance

$$\delta A_m = \partial_m\lambda_1 \quad \delta v_m = \partial_m\lambda_2 \quad (7)$$

Substituting the equations (2) and (5), into the master action lead to the action for the abelian topologically massive theory

$$I = \int d^3x[-\frac{1}{4}f^{mn}_{(A)}f_{mn(A)} - \frac{1}{4}\mu\epsilon^{mnp}A_m f_{np(A)}]. \quad (8)$$

On the other hand, we can eliminate the a_m and A_m fields, through the use of equations (5) and (6) in order to obtain

$$I = \int d^3x[-\frac{1}{2}\mu^2(v_m - \partial_m\phi)(v^m - \partial^m\phi) + \frac{1}{2}\mu\epsilon^{mnp}v_m\partial_n v_p], \quad (9)$$

which is invariant under the following abelian gauge transformations

$$\delta v_m = \partial_m\lambda_1, \quad \delta\phi = \lambda_1. \quad (10)$$

Fixing the gauge $\phi = 0$, we obtain the non-gauge invariant self-dual action. Then, the proposed master action show the equivalence (at classical level) between the topologically and self-dual theories. The master action that we are considering is locally equivalent to the master action of Deser and Jackiw, as can be seen after eliminating only the v_m field and is written down as

$$I = \int d^3x[-\mu\epsilon^{mnp}A_m\partial_n a_p - \frac{1}{2}\mu^2 a_m a^m - \frac{1}{2}\mu\epsilon^{mnp}A_m\partial_n A_p] \tag{11}$$

Introducing the Lie-algebra valued vectors $A_m = A^i_m T^i$, $a_m = a^i_m T^i$ and the Lie-algebra valued field strength $F_{mn} = F^i_{mn}T^i$, where the generators T^i of the gauge group are normalized by $T^iT^j = \delta^{ij}$, the non-abelian generalization of the master action of Deser and Jackiw obtained by replacing ordinary derivative by covariant derivative, $f_{mn} = \partial_m A_n - \partial_n A_m \rightarrow F_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n]$ and considering the non-abelian Chern-Simons term is

$$I = \mu tr \int d^3x[\epsilon^{mnp}a_m F_{np} - \frac{1}{2}\mu a_m a^m - \frac{1}{2}\epsilon^{mnp}A_m(\partial_n A_p + \frac{2}{3}A_n A_p)] \tag{12}$$

and only can reproduce the non-abelian version of the topologically massive theory after eliminating the a_m field by using its equation of motion ($a^m = \epsilon^{mnp}F_{np}$). On the other hand, the equation of motion obtained by independent variations in A_m has no known solutions and in consecuence the non-abelian master action of Deser and Jackiw can not reproduce the non-abelian self-dual action. The non-abelian topologically massive theory can be deduced from the self-interaction mechanism[14].

Now, we will consider for simplicity a triplet of $SU(2)$ free vector fields A^i_m coupled with a triplet of $SU(2)$ free vector fields v^i_m ($i = 1, 2, 3$). The action is

$$I_o = \int d^3x[-\mu\epsilon^{mnp}A^i_m\partial_n a^i_p - \frac{1}{2}\mu^2 a^i_m a^{mi} - \mu\epsilon^{mnp}A^i_m\partial_n v^i_p + \frac{1}{2}\mu\epsilon^{mnp}v^i_m\partial_n v^i_p]. \tag{13}$$

This action has two global simmetries. One is the global $SU(2)$ symmetry

$$\delta_\omega X = g\epsilon^{ijk}X^j\omega^k \tag{14}$$

where X = (A, a, v) and the other global symmetry is given by

$$\delta_\rho A^i_m = g\epsilon^{ijk}[a^j_m + v^j_m]\rho^k; \quad \delta_\rho a^i_m = 0 = \delta_\rho v^i_m. \tag{15}$$

Under these transformations, the action changes by a total derivative.

The Noether currents associated with the global symmetries are

$$j^{mi} = -\mu g\epsilon^{mnp}\epsilon^{ijk}A^j_n[a^k_p + v^k_p] + \frac{1}{2}\mu g\epsilon^{mnp}\epsilon^{ijk}v^j_n v^k_p \tag{16}$$

and

$$K^{mi} = -\frac{1}{2}\mu g\epsilon^{mnp}\epsilon^{ijk}[a^j_n + v^j_n][a^k_p + v^k_p]. \tag{17}$$

These currents are conserved on-shell. Now, we will couple these Noether currents to the action I_0 through the corresponding self-interaction term defined by

$$j^{mi} \equiv \frac{\delta I_{SI}}{\delta v^i_m}, \quad K^{mi} \equiv \frac{\delta I_{SI}}{\delta A^i_m}. \tag{18}$$

We find

$$\begin{aligned} I_{SI} &= g\mu \int d^3x[-\epsilon^{mnp}\epsilon^{ijk}v^i_m A^j_n a^k_p - \frac{1}{2}\epsilon^{mnp}\epsilon^{ijk}v^i_m v^j_n A^k_p \\ &- \frac{1}{2}\epsilon^{mnp}\epsilon^{ijk}A^i_m a^j_n a^k_p + \frac{1}{6}\epsilon^{mnp}\epsilon^{ijk}v^i_m v^j_n v^k_p]. \end{aligned} \tag{19}$$

The self-interaction mechanism stops here since no other derivative terms appear in I_{SI}. Now, we add I_{SI} to I_o. The last term in eq. (13) combines with the last term in eq. (19) to give a Chern-Simons term for the v_m field. The non-abelian action is

$$\begin{aligned} I &= \frac{1}{2}\mu \int d^3x[-\epsilon^{mnp}A^i_m(F^i_{np(a)} + F^i_{np(v)} + 2g\epsilon^{ijk}a_n v^k_p) - \mu a^i_m a^{mi} \\ &+ \epsilon^{mnp}v^i_m(\partial_n v^i_p + \frac{1}{3}\epsilon^{ijk}v^j_n v^k_p)], \end{aligned} \tag{20}$$

or

$$I = \frac{1}{2}\mu \int d^3x[-\epsilon^{mnp}A^i_m F^i_{np(a+v)} - \mu a^i_m a^{mi} + \epsilon^{mnp}v^i_m(\partial_n v^i_p + \frac{1}{3}\epsilon^{ijk}v^j_n v^k_p)], \tag{21}$$

where

$$F^i_{mn(a)} = \partial_m a^i_n - \partial_n a^i_m + g\epsilon^{ijk}a^j_m a^k_n \tag{22}$$

and

$$F^i_{mn(v)} = \partial_m v^i_n - \partial_n v^i_m + g\epsilon^{ijk}v^j_m v^k_n \tag{23}$$

are the field strengths for the a_m^i and v_m^i fields. The self-interaction process combines the abelian gauge transformations with the global ones giving rise to the following non-abelian local gauge transformations

$$\begin{aligned} \delta A_m^i &= g\epsilon^{ijk} A_m^j \alpha^k; \quad \delta a_m^i = g\epsilon^{ijk} a_m^j \alpha^k \\ \delta v_m^i &= \partial_m \alpha^i + g\epsilon^{ijk} v_m^j \alpha^k \end{aligned} \tag{24}$$

and

$$\begin{aligned} \delta A_m^i &= \partial_m \kappa^i + g\epsilon^{ijk}[a_m^j + v_m^j]\kappa^k \\ \delta a_m^i &= 0 = \delta v_m^i \end{aligned} \tag{25}$$

Defining $\omega_m \equiv a_m + v_m$, the action is rewritten down as

$$\begin{aligned} I &= \frac{1}{2}\frac{\mu}{g^2} tr \int d^3x[-\epsilon^{mnp} A_m F_{np(\omega)} - \mu(v_m - \omega_m)(v^m - \omega^m) \\ &+ \epsilon^{mnp} v_m[\partial_n v_p + \frac{2}{3} v_n v_p]. \end{aligned} \tag{26}$$

This action was interpreted as the interaction between a Chern-Simons and a BF(ϵAF) topological terms propagating a massive spin 1 physical mode[10]. Like as in the non-abelian topologically massive theory, invariance in the functional integral implies the quantization condition: $4\pi \frac{\mu}{g^2} = integer$.

We observe that A_m play the role of a Lagrange multiplier. Its equation of motion is

$$F_{mn(\omega)} = 0 \tag{27}$$

which tell us that ω is a pure gauge.

$$\omega_m = U^{-1}\partial_m U. \tag{28}$$

Then, the action becomes

$$I = \frac{1}{2}\frac{\mu}{g^2} tr \int d^3x[-\mu(v_m - U^{-1}\partial_m U)(v^m - U^{-1}\partial^m U) + \epsilon^{mnp} v_m(\partial_n v_p + \frac{2}{3} v_n v_p)], \tag{29}$$

where the v_m field appear coupled with a Stuckelberg field. Now, we have invariance under the following (finite) gauge transformations

$$v_m \to g^{-1}\partial_m \partial_m g + g^{-1} v_m g, \quad U \to Ug. \tag{30}$$

This gauge invariance allow us to fix the gauge $U = 1$, in order to obtain the standard action for the non-abelian self-dual field v_m

$$I = \frac{1}{2}\frac{\mu}{g^2} tr \int d^3[-\mu v_m v^m + \epsilon^{mnp} v_m(\partial_n v_p + \frac{2}{3} v_n v_p)]. \tag{31}$$

To conclude, we have derived the non-abelian self-dual action in three dimensions using the self-interaction mechanism. Recently, a dual version of a pure non-abelian Chern-Simons action was formulated [15]. It would be interesting to analyse the duality properties of the self-dual and topologically masive theories at non-abelian level.

ACKNOWLEDGEMENTS

The author would like to thank to Marti Ruiz Altaba for his hospitality at Instituto de Física de la Universidad Nacional Autónoma de México. Also, the author thanks Conicit-Venezuela for financial support.

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