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Energy–momentum types of warped spacetimes

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Abstract. The possible energy–momentum types of warped product spacetime are investigated using the classification given by Carot and da Costa.

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1. Introduction

Suppose that (M_1, h_1) and (M_2, h_2) are a pair of pseudo-Riemannian manifolds and θ is a real valued function on M_1. All structures will be considered smooth (where appropriate) and all manifolds Hausdorff, paracompact and connected. The *warped product* [2] of (M_1, h_1) and (M_2, h_2) with *warping factor* θ is the pseudo-Riemannian manifold with underlying manifold $M_1 \times M_2$ and metric g given by

$$g = \pi_1^* h_1 \otimes \mathrm{e}^{2\theta} \pi_2^* h_2. \tag{1}$$

In (1) the functions π_i are the canonical projections onto the factors of the product and will be omitted where there is no risk of confusion. The pseudo-Riemannian manifold (M, g) will be denoted by $M_1 \times_\theta M_2$. A *warped product spacetime*, or simply *warped spacetime*, is a four-dimensional manifold with a Lorentz metric, constructed in the above fashion. If the warping factor of a warped product spacetime is constant then the spacetime is (globally) decomposable, physically relevant examples of which are, for instance, the Bertotti–Robinson spacetimes [3, 4], or Einstein's static universe [22]. The class of warped product spacetimes with non-constant warping factors is, however, much richer and includes such well known examples as Schwarzschild, Friedmann–Robertson–Walker and static spacetimes. The purpose of this paper is to examine in detail which kinds of energy–momentum tensor are possible in warped product spacetimes and to provide some examples of such spacetimes.

Since the considerations in this work are mainly local it will be convenient to give a local version of the warped product definition. If $M_1 \times_\theta M_2$ is a warped product spacetime with $\dim M_i = n_i$ (for $i = 1, 2$) and $M = M_1 \times M_2$ then the convention will be adopted that upper-case Latin indices take values from 1 to n_1 and Greek indices take values from $n_1 + 1$ to 4, with lower-case Latin indices taking the values 1 to 4. For each $p \in M$ there

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exists a neighbourhood U of p such that there is a coordinate system x^a on U *adapted* to the product structure in the sense that the line element $\mathrm{d}s^2$ for M can be written in the form

$$\mathrm{d}s^2 = \underset{1}{h}_{AB}(x^A)\,\mathrm{d}x^A\,\mathrm{d}x^B + \mathrm{e}^{2\theta(x^A)}\,\underset{2}{h}_{\alpha\beta}(x^\alpha)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta. \tag{2}$$

Conversely, if a spacetime contains a neighbourhood U on which there is a coordinate system such that the line element takes the above form then it will be referred to as a *locally warped spacetime*. Some comments regarding the relationship between locally warped spacetimes and warped spacetimes will be made in the next section.

The importance of warped spacetimes is that their geometry is directly related to the geometry of their, lower-dimensional, factors which are generally easier to study. In the case where the first factor is Lorentzian the causal structure of the warped spacetime can be determined by examining the causal structure of the first factor [5, 6]. The warped product construction also provides a useful method for building examples of spacetimes with particular causal properties. Another aspect of the spacetime geometry, whose study is facilitated by the warped product structure, is the isometry algebra [1].

A classification of warped spacetime into three classes was given by Carot and da Costa [1], and the possible Petrov types determined in each case. These results will be reviewed here as they will be important later on. The three classes of warped products are denoted as A_1, A_2 and B and are defined as follows:

Class A_1. $n_1 = 1$ and $n_2 = 3$. These spacetimes are essentially characterized by the existence of a gradient conformal vector field (see next section) and if M_2 is Riemannian (positive definite) then the Petrov type is I, D or O, with no restrictions on the Petrov type in the case where M_2 is Lorentzian.

Class B. $n_1 = n_2 = 2$. These spacetimes are necessarily of Petrov type D or O and can be characterized by certain properties of the null directions tangent to the Lorentzian factor of the product (see the third section).

Class A_2. $n_1 = 3$ and $n_2 = 1$. These spacetimes are characterized by the existence of a hypersurface orthogonal non-null Killing vector and must be of Petrov type I, D or O if M_1 is Riemannian, with no restrictions on the Petrov type if M_1 is Lorentz.

It should be noted that the above three classes are not mutually exclusive (for example the Schwarzschild spacetime falls into classes A_2 and B).

Warped product spacetimes of class B have been considered previously by several authors [7–9], although they are not referred to as such. Some results on the relationship between warped product manifolds and certain pseudo-symmetry properties of the curvature tensor have been given by Deszcz *et al* [10, 11].

2. General geometric considerations

Certain geometric aspects of warped spacetimes will be discussed in this section and it is noted that, whilst the discussion is concerned with warped spacetimes, the considerations apply to warped product manifolds with arbitrary dimension and signature. The first point to be remarked upon is the distinction between warped products and locally warped products which was mentioned in the first section. The definition of a warped spacetime is essentially a global geometric one, yet if one is interested in exact solutions then one normally works locally and the local definition is more relevant. In fact a spacetime (M, g), which has the property that each $p \in M$ admits a neighbourhood U of p and an adapted coordinate system on U in which the line element takes the form (2), can be given the structure of a warped product under certain assumptions. Roughly speaking one has to assume that the local warped product structures on intersecting neighbourhoods are ‘compatible’, and that certain global geometric and topological restrictions hold. The existence of the (global) warped product structure then follows from a theorem of Ponge and Reckziegel [12] and a discussion of how this theorem is applied will be given elsewhere [13]. In [1] warped products of classes A_1 and A_2 were characterized by the existence of certain types of symmetry. The existence of these symmetries enables one to conclude that the metric can be written in the special form (2) and hence that the spacetime is a locally warped product. In fact it is possible to use direct geometric arguments to show that the existence of a warped product structure where one factor is one dimensional is equivalent to the existence of an appropriate type of symmetry [13].

In the case of a class A_1 warped product the local characterization given in [1] can be improved upon. If a spacetime (M, g) admits a (nowhere Killing) gradient conformal vector field X_a, i.e. X_a satisfies $X_{a;b} = \phi g_{ab}$ (where a semicolon will denote covariant differentiation and a comma partial differentiation) with ϕ a nowhere zero real-valued function, then the spacetime is locally warped of class A_1. From [1] it must be shown that X_a is non-null and $\phi_{,a}$ is parallel to X_a in order to conclude that (M, g) is locally a warped product. However the inequality $X^a X_{a;b} \neq 0$ shows that $X^a X_a$ cannot vanish over an open set and an application of the Ricci identity gives

$$R_{abcd} X^d = 2\phi_{,[a} g_{b]c}. \tag{3}$$

In the above, and in what follows, R_{abcd} denotes the Riemann curvature tensor, square brackets denote skew-symmetrization and round brackets denote the symmetrization operation. If (3) is contracted with X^b then one obtains $\phi_{[,a} X_{b]} = 0$ showing that $\phi_{,a}$ is parallel to X_a and hence (M, g) is locally a warped product. The degenerate case where ϕ is identically zero and X_a is non-null corresponds to the special case of a locally decomposable spacetime or, equivalently, a warped product with constant warping factor.

Finally in this section it will be shown that all warped spacetimes admit a (second-order) Killing tensor, that is a second-order symmetric tensor K_{ab} satisfying $K_{(ab;c)} = 0$. It should be remarked, however, that this Killing tensor may be decomposable, that is, expressible in terms of Killing vectors. Suppose that (M, g) is a warped spacetime and that equation (1) is written in local coordinates as

$$g_{ab} = \underset{1}{h}_{ab} + \mathrm{e}^{2\theta} \underset{2}{h}_{ab}. \tag{4}$$

The spacetime with metric $g'_{ab} = \mathrm{e}^{-2\theta} g_{ab}$ is decomposable and so $\underset{2}{h}_{ab}$ is covariantly constant under the Levi–Civita connection associated with g'_{ab}. Using the well known formula for the change in connection under a conformal change of metric one has

$$\underset{2}{h}_{ab;c} = -2 \underset{2}{h}_{ab} \theta_{,c} - \underset{2}{h}_{bc} \theta_{,a} - \underset{2}{h}_{ac} \theta_{,b}. \tag{5}$$

It then follows that $K_{ab} \equiv \mathrm{e}^{4\theta} \underset{2}{h}_{ab}$ is a Killing tensor and hence one has the following theorem.

Theorem 1. Suppose that (M, g) is a warped spacetime. It then follows that (M, g) admits a second-order Killing tensor which is proportional to the metric of the second factor of the product.

3. Warped spacetimes of class B

In this section a thorough study of class B warped spacetimes will be made and their possible energy–momentum types determined. A theorem will be proven which shows that in many cases the second factor of the warped product is of constant curvature, and a geometric proof of an 'extended Birkhoff theorem' will be given.

If (M, g) is a warped spacetime of class B then one of the factors is a two-dimensional Lorentz manifold which necessarily admits a unique pair of independent null one-dimensional distributions. A pair of null vectors can locally be chosen to span these distributions and then lifted to the spacetime to give null vectors l^a and n^a whose directions are canonically determined by the product structure. A warped product spacetime of class B can, in fact, be characterized by properties of these null directions [1]. Under the decomposable metric g' which is conformally related to g the vectors l^a and n^a are recurrent in the sense that there exist 1-forms p_a and q_a such that $l_{a|b} = l_a p_b$ and $n_{a|b} = n_a q_b$ (where covariant differentiation with respect to the decomposable metric is denoted by a stroke). These null vectors are therefore geodesic, shearfree and twistfree under g' and hence under all metrics conformally related to g'. In fact the g-covariant derivatives of l_a and n_a satisfy

$$l_{a;b} = l_a p_b - \theta_{,a} l_b + (l^c \theta_{,c}) g_{ab} \tag{6}$$

$$n_{a;b} = n_a q_b - \theta_{,a} n_b + (n^c \theta_{,c}) g_{ab}. \tag{7}$$

In the above one has that $\theta_{[,a} l_b n_{c]} = 0$ when M_1 is Lorentzian and $l^a \theta_{,a} = n^a \theta_{,a} = 0$ when M_2 is Lorentzian. Conversely if some spacetime admits a pair of independent null directions l_a and n_a satisfying (6) and (7) for some $\theta_{,a}$ which either lies in the l–n plane or its orthogonal complement at each point then this spacetime is locally warped [1].

A classification of class B warped spacetimes into four subclasses will now be given, depending on the relationship between the gradient of the warping factor and the pair of null directions picked out by the product structure. There is an obvious classification into two types depending on whether M_1 is Lorentzian or Riemannian but it will prove convenient to further subdivide the first type, and introduce a degenerate type (corresponding to the decomposable case). Suppose that (M, g) is a class B warped spacetime $M_1 \times_\theta M_2$ and that l_a and n_a span the null directions canonically determined by the product structure. This spacetime is then classified as follows.

B_T. If $\theta_{,a} \neq 0$, $l_{[a} n_b \theta_{,c]} = 0$, $\theta_{[,a} l_{b]} \neq 0$ and $\theta_{[,a} n_{b]} \neq 0$ then the warped spacetime will be said to be of type B_T. In this case M_1 is Lorentzian.

B_R. If $\theta_{,a} \neq 0$, $l_{[a} n_b \theta_{,c]} = 0$ and either $\theta_{[,a} l_{b]} = 0$ or $\theta_{[,a} n_{,b]} = 0$ (but not both) then the warped spacetime will be said to be of type B_R. Without loss of generality it will be assumed that spacetimes of this type have $\theta_{,b}$ parallel to l_b. This class of type B warped products may alternatively be characterized by $\theta_{[,a} l_b n_{c]} = 0$ and $l^a \theta_{,a} = 0$. Using the standard expression for the change in connection under a conformal change of metric it may be shown that l^a is recurrent in this case.

B_S. If $\theta_{,a} \neq 0$ and $l^a \theta_{,a} = n^a \theta_{,a} = 0$ then the warped product spacetime will be said to be of type B_S. In this case M_2 is Lorentzian.

B_P. If $\theta_{,a} = 0$ then the spacetime will be said to be of type B_P and this is the decomposable case.

The above classification is not mutually exclusive as one could construct spacetimes which fall into different types on different regions. Since a class B warped product either has M_1 Lorentzian or Riemannian, such a spacetime cannot be both type B_S and B_T on separate non-empty subsets. However a class B warped spacetime (M, g), with the first factor Lorentzian, can be decomposed as a disjoint union $M_T \cup M_R \cup M_P \cup M'$ where the spacetime is of type B_T on the open set M_T; M_R and M_P are the interiors of the subsets of type B_R and B_P, respectively, and M' is defined by the decomposition. An analogous decomposition is possible in the case where the first factor is Riemannian. Consequently one can always work in an open submanifold where one of the above four types holds exclusively, and from now on it will be assumed that one is working in such a submanifold. It will also be convenient to restrict oneself to an open submanifold where the Segre type of the Ricci tensor is constant. It is remarked that in types B_S and B_T neither l^a nor n^a are recurrent, in type B_R exactly one of the two vectors is recurrent and in type B_P both are recurrent.

The Ricci tensor of a class B warped spacetime will now be examined in more detail and the relationship between the Segre type and the above classification determined. An adapted coordinate system of the form described in the introduction will generally be used and, unless otherwise stated, indices will be raised and lowered with the spacetime metric g_{ab}. If the Ricci tensors associated with $\underset{1}{h}_{ab}$ and $\underset{2}{h}_{ab}$ are denoted by $\underset{1}{R}_{ab}$ and $\underset{2}{R}_{ab}$ respectively then the Ricci tensor R_{ab} associated with g_{ab} is given by the following [1, 6]

$$R_{AB} = \underset{1}{R}_{AB} - \frac{2}{\phi}\phi_{;AB} \tag{8}$$

$$R_{\alpha\beta} = \underset{2}{R}_{\alpha\beta} - \tfrac{1}{2}(\phi^2)^A{}_{;A}\,\underset{2}{h}_{\alpha\beta} \tag{9}$$

$$R_{\alpha B} = 0. \tag{10}$$

In (8)–(10), ϕ is defined by $\phi = \mathrm{e}^{\theta}$, where θ is the warping factor. The block-diagonal form of the Ricci tensor and metric in the adapted coordinate system means that the Segre type can be determined fairly easily from the Segre type of the pair of 2×2 matrices R_{AB} and $R_{\alpha\beta}$. This is because one can reduce each block to Jordan form using commuting similarity transformations. In fact, since the Ricci tensor of a two-dimensional manifold is a multiple of the metric it follows from (9) that the Segre type of $R_{\alpha\beta}$ is just $\{(11)\}$ (or $\{(1,1)\}$ if M_2 is Lorentzian). One simply needs to compute the Segre characteristic of R_{AB} and 'adjoin' it to the Segre characteristic of $R_{\alpha\beta}$. If M_1 is Riemannian, then the Segre type of R_{AB} is just $\{11\}$ (or its degeneracy) but if M_1 is Lorentzian then the three Segre types $\{2\}$, $\{z\bar{z}\}$ and $\{11\}$ are possible. In the case of type B_R it follows from the existence of a null recurrent vector and the Ricci identities that R_{ab} has a null eigenvector l^a satisfying $l^\alpha = 0$. For type B_P one has that ϕ is a constant. The possible Ricci types corresponding to the four types of class B warped spacetime just described are given in table 1. Note that all degeneracies of the types given in the table are also possible.

Table 1. Energy–momentum types of class B warped products.

Warped type	Segre type
B_T	$\{2(11)\}$, $\{z\bar{z}(11)\}$ or $\{1,1(11)\}$
B_R	$\{2(11)\}$ or $\{(1,1)(11)\}$
B_S	$\{(1,1)11\}$
B_P	$\{(1,1)(11)\}$

Given the above classification, the question arises as to whether examples actually exist. Since all spherically symmetric spacetimes are warped products where the first factor is Lorentzian [1] (and the second factor is of constant curvature), such examples are easily found in the literature. Specifically, the Schwarzschild and Reissner–Nordström solutions are of type B_T; type B_R spherically symmetric solutions have been given by Foyster and McIntosh [14] and the Bertotti–Robinson solutions mentioned in the introduction are of type B_P. Examples of spacetimes of type B_S are less easy to find in the literature and it can be seen from the table that the only physically interesting energy–momentum types possible are non-null electromagnetic, Λ-term or vacuum. It will be shown shortly that all these solutions necessarily admit a three-dimensional isometry algebra acting on two-dimensional timelike orbits (i.e. the second factor of the product is of constant curvature). A particular example of a vacuum solution of type B_S is given by the following line element, defined on an appropriate open subset of $\mathbb{R}^4$ with coordinates (x, y, z, t)

$$\mathrm{d}s^2 = F^{-1}\,\mathrm{d}x^2 + F\,\mathrm{d}y^2 + x^2(\sin^2(t)\,\mathrm{d}z^2 - \mathrm{d}t^2) \tag{11}$$

where

$$F(x) \equiv \frac{A}{x} - 1 \qquad A \text{ constant.} \tag{12}$$

It has been shown previously that the second factor of a class B warped spacetime is of constant curvature in the case of vacuum [8] and perfect fluid [9]. It then follows that the Killing vector algebra of the second factor is three dimensional and all these Killing vectors can be lifted to M [1]. In fact this conclusion can be reached for a wide variety of energy–momentum types and the following is an attempt at formulating a theorem which applies to most cases of interest. Note that the product structure can be used to define a vector (at a point) X^a to be *vertical* if $\underset{1}{h}_{ab}X^a = 0$ and *horizontal* if $\underset{2}{h}_{ab}X^a = 0$, and that Ricci eigenvectors must be either horizontal or vertical, at each point.

Theorem 2. Suppose that (M, g) is a class B warped product spacetime with factors $M_1 \times_\theta M_2$ and warping factor θ. In addition suppose that, at each point in M, the subspace of the cotangent space spanned by the gradients of the eigenvalues associated with all Ricci eigenvectors can be spanned by the gradients of the eigenvalues associated with horizontal eigenvectors. It then follows that M_2 is necessarily of constant curvature.

Corollary 1. The conclusions of the theorem apply in the case of vacuum, Λ-term, electromagnetic (null and non-null) and perfect fluid (with an equation of state).

Proof. Denote the Ricci scalars associated with (M_1, h_1) and (M_2, h_2) by $\underset{1}{R}$ and $\underset{2}{R}$, respectively. One can then rewrite the field equations (8) and (9) as

$$R_{AB} = \frac{1}{2}\,\underset{1}{R}\,\underset{1}{h}_{AB} - \frac{2}{\phi}\phi_{;AB} \tag{13}$$

$$R_{\alpha\beta} = \tfrac{1}{2}\,\underset{2}{R}\,\underset{2}{h}_{\alpha\beta} - \tfrac{1}{2}(\phi^2)^{A}{}_{;A}\,\underset{2}{h}_{\alpha\beta}. \tag{14}$$

Firstly, suppose that there are non-constant Ricci eigenvalues. From the considerations on Segre type earlier in this section it follows that there exists at least one horizontal Ricci eigendirection Z^a and from the above assumption its eigenvalue λ may be assumed to be non-constant. Now, from (13) and (14) (block-diagonal structure of the Ricci tensor) it follows that λ must be a root of the characteristic polynomial associated with $R^A{}_B$ and therefore (see (13)) a function on M_1; that is it only depends on the coordinates x^1 and x^2, and the same holds for its associated (horizontal) eigendirection Z^a, that is: Z^a is tangent to M_1 and its components are functions on M_1. Now from (14) it is immediately seen that the (degenerate) vertical eigenvalue μ is given by

$$\mu = \tfrac{1}{2}\phi^{-2}(\underset{2}{R} - (\phi^2)^{A}{}_{;A}) \tag{15}$$

and, according to the hypotheses of the theorem, it must be a (possibly constant) function of x_1 and x_2. Since μ and ϕ are functions of the first two coordinates only and $\underset{2}{R}$ is a function of x^3 and x^4 only it follows from the above that $\underset{2}{R}$ is a constant. In the case where all Ricci eigenvalues are constant, one arrives at equation (15) with μ constant and can again conclude that $\underset{2}{R}$ is constant. This completes the proof of the theorem. □

The corollary follows immediately in the case of vacuum, Λ-term and electromagnetic null since in these cases all the Ricci eigenvalues are constant. In the case of electromagnetic non-null the tracefree condition shows that the pair of eigenvalues are functionally related (and one of them is associated with a horizontal eigenvector) and in the case of a perfect fluid the existence of an equation of state similarly shows that the conditions of the theorem hold.

Finally in this section a geometric proof of an ‘extended Birkhoff theorem’ will be given. Other proofs of extended Birkhoff theorems have been given by Goenner [15] and Barnes [16] and a more geometrical proof was given by Bona [17].

Theorem 3. Suppose that (M, g) is a class B warped product spacetime whose Ricci tensor is everywhere of Segre type $\{(1, 1)(11)\}$ or some degeneracy thereof. Assume that the gradient of the warping factor does not vanish over a non-empty open set. It then follows that one can decompose M as the disjoint union $M = M_K \cup M_R \cup M'$ where M_K and M_R are open and a hypersurface orthogonal non-null Killing vector is admitted on a neighbourhood of any point of M_K and a null recurrent vector field admitted on a neighbourhood of any point of M_R. The set M' has no interior.

Corollary 2. For non-flat (in the sense that the curvature does not vanish over a non-empty open subset) Einstein spaces one has that $M_R = M' = \phi$.

Proof. Given the assumptions on Segre type, one can write the Ricci tensor as $R_{ab} = \sigma \underset{1}{h}_{ab} + \rho \underset{2}{h}_{ab}$ (for some functions σ and ρ) and hence the field equations (8) and (9) become:

$$\sigma \underset{1}{h}_{AB} = \frac{1}{2} \underset{1}{R} \underset{1}{h}_{AB} - \frac{2}{\phi}\phi_{;AB} \tag{16}$$

$$\rho \underset{2}{h}_{\alpha\beta} = \tfrac{1}{2} \underset{2}{R} \underset{2}{h}_{\alpha\beta} - \tfrac{1}{2}(\phi^2)^{A}{}_{;A} \underset{2}{h}_{\alpha\beta}. \tag{17}$$

It can be seen from the above that M_1 admits a gradient conformal vector ϕ_A, and if this is non-null (as it must be in the B_S and B_T cases, but not in the B_R case) then M_1 is locally a 1+1 warped product (see remarks in section 2 and [13]). The factor M_1 then admits (locally) a hypersurface orthogonal Killing vector which is orthogonal to the gradient of the warping factor and so [1] lifts up to give a hypersurface orthogonal Killing vector on M. Actually the field equations (16) and (17) reduce to the vacuum or Λ-term case if one has σ and ρ zero or constant and the conclusion about the existence of a hypersurface orthogonal Killing vector follows in this case also. In any non-empty open subset of M of type B_R, the ‘extra’ Killing vector is not necessarily admitted, but a null recurrent vector is admitted. This completes the proof of the theorem. The corollary follows from the fact that a non-flat Einstein space admitting a recurrent vector is necessarily of Petrov type N or III [18]. □

4. Warped spacetimes of class A_1

The purpose of this section is to determine what sort of energy–momentum tensors are possible in warped spacetimes of class A_1. It will be shown that vacuum and Λ-term types imply that the spacetime must be of constant curvature, the only possible solution then being the De Sitter spacetimes (including Minkowski, which is trivially seen to be decomposable, i.e. the warping function is constant), whereas non-null electromagnetic fields are not possible. On the other hand, examples will be given of null electromagnetic and perfect fluid solutions which are warped products of class A_1. In the latter two cases some general remarks can be made concerning the possible Petrov types.

As was remarked upon in section 2, class A_1 warped products can essentially be characterized by the existence of a (non-Killing) gradient conformal vector field. Some results concerning spacetimes admitting gradient conformals have been given by Tariq and Tupper [19] and Daftardar and Dadhich [20]. In addition, class A_1 warped spacetimes with the second factor Riemannian have recently been referred to as *generalized Robertson–Walker spacetimes* [21].

For later reference it will be convenient to display the expressions for the Ricci tensor [1, 6] specialized to a class A_1 warped spacetime (M, g) with warping factor θ. As before an adapted coordinate system will be used and $\phi \equiv \mathrm{e}^{\theta}$.

$$R_{AB} = -\frac{3}{\phi}\phi_{;AB} \tag{18}$$

$$R_{\alpha\beta} = \underset{2}{R}_{\alpha\beta} + \{-2(\phi_C\phi^C) - \phi\phi^C{}_{;C}\}\,\underset{2}{h}_{\alpha\beta}. \tag{19}$$

If ordinary differentiation with respect to x^1 is denoted by a prime then the above can be rewritten as

$$R_{11} = -\frac{3}{\phi}\phi'' \tag{20}$$

$$R_{\alpha\beta} = \underset{2}{R}_{\alpha\beta} + \epsilon\{-2(\phi')^2 - \phi\phi''\}\,\underset{2}{h}_{\alpha\beta} \qquad \epsilon = g_{11} = \pm 1. \tag{21}$$

It can be seen from the above that the vector $\partial/\partial x^1$ is necessarily an eigenvector of the Ricci tensor. If this vector is spacelike (i.e. if M_2 is Lorentzian) then no restrictions are placed on the Segre type of the Ricci tensor, but if this is timelike (i.e. M_2 is Riemannian) then the Ricci tensor is necessarily diagonalizable with $\partial/\partial x^1$ spanning a timelike eigendirection.

If one has that $R_{ab} = Kg_{ab}$ for some (perhaps zero) constant K then equation (21) shows that $\underset{2}{R}_{ab}$ is a multiple of the metric on M_2. The Bianchi identity on M_2 then shows that M_2 is a space of constant curvature and hence M admits a six-dimensional isometry algebra acting on three-dimensional orbits. However, it then follows that at each point of M, the Killing isotropy group is three dimensional and hence the Weyl tensor must vanish at each point [22], the resulting spacetime thus being of constant curvature. The following theorem has therefore been established (cf [11]), which effectively says that the only vacuum or Λ-term solutions which are locally warped products of class A_1 are the De Sitter spacetimes.

Theorem 4. Let (M, g) be a warped product of class A_1. If $R_{ab} - \frac{1}{4}Rg_{ab} = 0$ over some non-empty open set U then it follows that the curvature is constant on U.

Suppose now that the Ricci tensor given by (20) and (21) corresponds to a non-null electromagnetic field, its canonical Segre form therefore being [23]

$$R_{ab} = \psi^2\{x_a x_b + y_a y_b - z_a z_b + u_a u_b\} \tag{22}$$

where $\{x^a, y^a, z^a, u^a\}$ are unit eigenvectors that form an orthonormal tetrad with corresponding eigenvalues ψ^2 for x^a and y^a and $-\psi^2$ for z^a and u^a. As we have already pointed out, $\partial/\partial x^1$ is a (unit) Ricci eigenvector (timelike for $\epsilon = -1$, spacelike for $\epsilon = +1$) which can then be identified with one of the above vectors in the tetrad, say x^a if $\epsilon = +1$ or u^a if $\epsilon = -1$. It therefore follows that

$$\psi^2 = -\frac{3}{\phi}\phi'' \tag{23}$$

and the three remaining eigenvectors must necessarily lie in M_2, two of them (say y^a and z^a) being spacelike and having opposite sign eigenvalues $+\psi^2$ and $-\psi^2$. Using (21) to obtain the eigenvalue equation for this pair of eigenvectors one has

$$\{\underset{2}{R}_{\alpha\beta} - \epsilon(2\phi'^2 + \phi\phi'')\underset{2}{h}_{\alpha\beta} \pm 3\phi\phi''\underset{2}{h}_{\alpha\beta}\}\omega^\beta_\pm = 0 \tag{24}$$

where $\omega^\beta_+ \equiv y^\beta$ and $\omega^\beta_- \equiv z^\beta$. Hence $\lambda_\pm \equiv (\pm 3 - \epsilon)\phi\phi'' - 2\epsilon\phi'^2$ must both be roots of the characteristic polynomial associated with $\underset{2}{R}_{\alpha\beta}$, and since $\phi = \phi(x^1)$ both of them must be constants, which immediately implies that ϕ' is constant; that is $\phi'' = \psi^2 = 0$. Thus, no non-null Einstein–Maxell warped spacetimes of the class A_1 exist.

It will next be assumed that (M, g) is a warped spacetime of class A_1 whose Ricci tensor is everywhere of Segre type $\{(211)\}$ with constant eigenvalue. That is to say that the spacetime represents a null electromagnetic field or null radiation field with a possibly non-zero cosmological constant. Examples with non-zero cosmological constant have been given by Tariq and Tupper [19] and these will be discussed shortly but first some general remarks about the Petrov type will be made.

Since the Ricci tensor has no timelike eigenvalues it follows that M_2 is Lorentzian and the gradient conformal X_a is spacelike. At an arbitrary point $p \in M$ let l_a, n_a, x_a, y_a be a real null tetrad where l_a is the (unique up to scaling) null Ricci eigenvector and x_a is parallel to X_a. Equation (3) and the subsequent remarks then show that there exists a scalar α such that

$$R_{abcd}x^d = \alpha x_{[a}g_{b]c}. \tag{25}$$

Now consider the decomposition (see e.g. [22])

$$R_{abcd} = C_{abcd} + E_{abcd} + \tfrac{1}{6}Rg_{a[c}g_{d]b} \tag{26}$$

where in the above the following definitions have been used:

$$E_{abcd} = S_{a[c}g_{d]b} - S_{b[c}g_{d]a} \tag{27}$$

$$S_{ab} = R_{ab} - \tfrac{1}{4}Rg_{ab}. \tag{28}$$

It follows from the assumptions concerning Segre type that [23]

$$E_{abcd}x^c y^d = E_{abcd}x^c l^d = 0. \tag{29}$$

Hence from (25), (26) and (29) and the fact that $C_{abcd}F^{cd} = \lambda F_{ab}$ is equivalent to $C_{abcd}\overset{*}{F}{}^{cd} = \lambda\overset{*}{F}_{ab}$ (where an asterisk denotes the usual dual operation) one has

$$C_{abcd}l^c x^d = \lambda l_{[a}x_{b]} \qquad C_{abcd}l^c y^d = \lambda l_{[a}y_{b]} \qquad C_{abcd}l^c n^d = \lambda l_{[a}n_{b]} \tag{30}$$

for some λ. Equation (30) implies that

$$C_{abcd}l^c = \lambda(x_d l_{[a}x_{b]} + y_d l_{[a}y_{b]} + l_d l_{[a}n_{b]}). \tag{31}$$

Transvecting with g^{ad} then shows that $\lambda = 0$ and so $C_{abcd}l^d = 0$. One can therefore conclude that if the Weyl tensor is non-zero at p, it is of Petrov type N with repeated principal null direction l_a. The following theorem has been established (cf [24]).

Theorem 5. Suppose that (M, g) is a warped product spacetime of class A_1 with Ricci tensor of Segre type $\{(211)\}$ at each point. It then follows that the Petrov type is everywhere either N or O.

Examples of class A_1 warped spacetimes representing null Einstein–Maxwell fields, both with and without cosmological constant, will now be discussed. Suppose that $R_{ab} = l_a l_b + K g_{ab}$, where K is a constant (possibly zero). The field equations (20), (21) then become

$$K = -\frac{3}{\phi}\phi'' \tag{32}$$

$$K\phi^2 \underset{2}{h}_{\alpha\beta} + l_\alpha l_\beta = \underset{2}{R}_{\alpha\beta} + \{-2(\phi')^2 - \phi\phi''\} \underset{2}{h}_{\alpha\beta}. \tag{33}$$

The gradient conformal X_a has components $(\phi, 0, 0, 0)$ in the adapted coordinate system and satisfies $X_{a;b} = \phi' g_{ab}$. In the case of non-zero cosmological constant, a judicious choice of solution ϕ to the ordinary differential equation (32) enabled Tariq and Tupper to produce a family of solutions [19]. In fact, if one sets $\phi = \mathrm{e}^{Rx}$ where $-R^2 = K/3$ then (33) reduces to

$$l_\alpha l_\beta = \underset{2}{R}_{\alpha\beta}. \tag{34}$$

An example of a three-dimensional Lorentz manifold satisfying (34) is the three-dimensional analogue of the pp-waves of Ehlers and Kundt [25] and the full four-dimensional line element is then given by

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{e}^{2Rx}(\mathrm{d}y^2 - 2H(u, y)\,\mathrm{d}u^2 - 2\mathrm{d}u\,\mathrm{d}v). \tag{35}$$

In the above, the coordinates (x, y, u, v) are adapted to the warped product structure and the third partial derivative of H with respect to y vanishes identically [19].

If the cosmological constant is assumed to be zero then it follows from equation (32) that $\phi'' = 0$ and so, in this case, X_a is actually a homothety. It can then be shown [24] that the metric g belongs to the class known as 'Kundt's class' (see e.g. [22]) and the line element is given locally by

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - 2\mathrm{d}u\,\mathrm{d}v + \frac{4v}{x}\,\mathrm{d}u\,\mathrm{d}x + ((v^2/x^2) - 2G(u, x, y))\,\mathrm{d}u^2 \tag{36}$$

where it should be noted that the coordinate system (x, y, u, v) in (36) is *not* adapted to the warped product structure. Imposing the condition that a gradient homothety is admitted then shows that the function G takes the form $G = xyf(u, x/y)$ for an arbitrary function f (which is nonlinear in x/y otherwise the solution is vacuum and hence flat). The gradient homothety then has covariant components $(x, y, 0, 0)$ in the (x, y, u, v) coordinate system.

The last case to be considered in this section is the perfect fluid case. Assume that a global unit timelike vector field u_a and global functions μ and p exist such that the Ricci tensor can be written as

$$R_{ab} = (\mu + p)u_a u_b + \frac{\mu - p}{2} g_{ab}. \tag{37}$$

It will also be assumed that an equation of state $\mu = \mu(p)$ holds and that $\mu + p$ is nowhere vanishing. In the case where M_2 is Riemannian the vector $\partial/\partial x^1$ (in an adapted coordinate system) spans the unique timelike Ricci eigendirection and so can be assumed to be equal to u^a. On substituting (37) into the second field equation (21) one finds that $\underset{2}{R}_{\alpha\beta}$ is a multiple of $\underset{2}{h}_{\alpha\beta}$ and hence M_2 is of constant curvature. It has therefore been shown that the only perfect fluid warped spacetimes of class A_1 with the second factor Riemannian are the Friedmann–Robertson–Walker spacetimes (cf [1, 20]).

Now assume that M_2 is Lorentzian and that an adapted coordinate system x^a is being used. In these coordinates the gradient conformal X_a is given by $(\phi, 0, 0, 0)$ and satisfies $X_{a;b} = \phi' g_{ab}$. The field equations (20) and (21) in this case (noting that $u_1 = 0$) become

$$\frac{\mu - p}{2} = -\frac{3}{\phi}\phi'' \tag{38}$$

$$\phi^2 \frac{\mu - p}{2} \underset{2}{h}_{\alpha\beta} + (\mu + p)u_\alpha u_\beta = \underset{2}{R}_{\alpha\beta} + \{-2(\phi')^2 - \phi\phi''\} \underset{2}{h}_{\alpha\beta}. \tag{39}$$

On differentiating equation (38) with respect to x^α, for some α, one obtains the following

$$\frac{\mathrm{d}\mu}{\mathrm{d}p}\frac{\partial p}{\partial x^\alpha} = \frac{\partial p}{\partial x^\alpha}. \tag{40}$$

It can then be seen that either $\mathrm{d}\mu/\mathrm{d}p = 1$ or μ and p are both functions of x^1 only. The equation (39) can be considered as an equation for the Ricci tensor in M_2 and if $\tilde{u}_\alpha$ is the projection of u_a to M_2, scaled so as to be unit with respect to $\underset{2}{h}_{\alpha\beta}$, one has

$$\underset{2}{R}_{\alpha\beta} = \{-\phi\phi'' + 2(\phi')^2\} \underset{2}{h}_{\alpha\beta} + \phi^2(\mu + p)\tilde{u}_\alpha \tilde{u}_\beta. \tag{41}$$

Now, as it can readily be seen from the above expression, $\underset{2}{R}_{\alpha\beta}$ is of the Segre type $\{1, (11)\}$ with eigenvalues $\lambda_1 \equiv 2(\phi'^2 - \phi\phi'') - (\mu + p)\phi^2$ and $\lambda_2 = \lambda_3 \equiv 2(\phi'^2 - \phi\phi'')$ where λ_1 is the eigenvalue associated to the (unit) timelike eigenvector $\tilde{u}^\alpha$. Since λ_i $(i = 1, 2, 3)$ must all be roots of the characteristic polynomial of $\underset{2}{R}{}^\alpha_\beta$ it follows that they are all functions of the coordinates (x^γ), and since $\phi = \phi(x^1)$ this in turn implies: $2(\phi'^2 - \phi\phi'') \equiv \lambda_0$ with λ_0 = constant, and $(\mu + p)\phi^2 \equiv S(x^\gamma)$, thus the above equation can be rewritten as

$$\underset{2}{R}_{\alpha\beta} = S(x^\gamma)\tilde{u}_\alpha \tilde{u}_\beta + \lambda_0 \underset{2}{h}_{\alpha\beta} \tag{42}$$

with

$$\mu + p = \frac{S(x^\gamma)}{\phi^2} \qquad \mu - p = -6\frac{\phi''}{\phi}. \tag{43}$$

Furthermore, the condition that $\left(\phi'^2 - \phi\phi''\right)' = 0$ is equivalent to

$$\frac{\phi''}{\phi} = \text{constant} \equiv k \tag{44}$$

and therefore

$$\mu = \frac{S(x^\gamma)}{2\phi^2} - 3k \qquad p = \frac{S(x^\gamma)}{2\phi^2} + 3k. \tag{45}$$

Notice that the energy conditions $\mu > 0$ and $\mu \pm p \geqslant 0$ are satisfied provided $S(x^\gamma) \geqslant 0$ and $k \leqslant 0$.

Solutions of these characteristics do indeed exist as the following example (for which $k = 0$, that is $p = \mu$) shows:

$$ds^2 = dx^2 + x^2(-dt^2 + F^2(t, y, z)\, dy^2 + H^2(t, y, z)\, dz^2) \tag{46}$$

where

$$F = \frac{1}{H_0}\left(C_1 e^{2t} + C_2 e^{-2t} - \frac{R}{4}\right)^{1/2} \exp\left\{\frac{4C_0}{(64C_1C_2 - R^2)^{1/2}} \arctan\left(\frac{8C_2 e^{-2t} - R}{(64C_1C_2 - R^2)^{1/2}}\right)\right\} \tag{47}$$

$$H = \frac{1}{F}(C_1 e^{2t} + C_2 e^{-2t} - R/4) \tag{48}$$

C_1, C_2, R, H_0 and C_0 all being functions of y and z that must satisfy the following two equations:

$$\frac{H_{,ty}}{H_{,y}} = \frac{C_1 e^{2t} - C_2 e^{-2t} - C_0}{C_1 e^{2t} + C_2 e^{-2t} - \frac{R}{4}} \qquad \frac{F_{,tz}}{F_{,z}} = \frac{C_1 e^{2t} - C_2 e^{-2t} + C_0}{C_1 e^{2t} + C_2 e^{-2t} - \frac{R}{4}}. \tag{49}$$

The following is a family of solutions of the above type:

$$ds^2 = -x^2\, dt^2 + dx^2 + x^2 \sinh(2t)[(\tanh(t))^{-q}\, dy^2 + (\tanh(t))^q\, dz^2] \tag{50}$$

where $q \in [0, 1)$. The density is then

$$\mu = \frac{(1 - q^2)}{x^2 \sinh^2(2t)}. \tag{51}$$

In the case where the equation of state takes the form $\mu = p$ the first field equation (38) shows that $\phi'' = 0$ and hence X_a is a homothety. The Ricci identity for X_a then implies that $R_{abcd}X^a = 0$ and this will enable information about the Petrov type to be deduced. Fix a point $p \in M$ and an orthonormal tetrad u_a, x_a, y_a, z_a where u_a is as before and x_a is parallel to X_a. The Segre type of the Ricci tensor then implies that $x_{[a}y_{b]}$ and $x_{[a}z_{b]}$ are eigenbivectors of the tensor E_{abcd} (defined by (27)) with equal eigenvalues [23]. These bivectors also satisfy $R_{abcd}x^{[a}z^{b]} = R_{abcd}x^{[a}y^{b]} = 0$ and hence from equation (26) they are eigenbivectors of the Weyl tensor with equal eigenvalues. Using the property $C_{abcd}F^{cd} = \lambda F_{ab} \Leftrightarrow C_{abcd}\overset{*}{F}{}^{cd} = \lambda \overset{*}{F}{}_{ab}$ of the Weyl tensor one therefore has (for some $\lambda \in \mathbb{R}$):

$$\begin{aligned} C_{abcd}u^c y^d &= \lambda u_{[a}y_{b]} \qquad & C_{abcd}u^c z^d &= \lambda u_{[a}z_{b]} \\ C_{abcd}x^c z^d &= \lambda x_{[a}z_{b]} \qquad & C_{abcd}x^c y^d &= \lambda x_{[a}y_{b]}. \end{aligned} \tag{52}$$

Now define $l_a = x_a + u_a$ and $n_a = x_a - u_a$ and then using equations (52) one obtains, after some calculation,

$$C_{abcd}l^b l^c y^d = 0 = C_{abcd}l^b l^c z^d \Rightarrow l^b l^c C_{abc[d}l_{e]} = 0 \tag{53}$$

and an analogous equation involving n_a. These equations express the fact that l^a and n^a span repeated principal null directions of the Weyl tensor and hence it is (if non-zero) of Petrov type D. It is also noted that the principal null directions lie in the timelike 2-plane spanned by u_a and X_a at each point.

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