Curvature singularity of the distributional BTZ black hole geometry

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For the non-rotating BTZ black hole, the distributional curvature tensor field is found. It is shown to have singular parts proportional to a δ-distribution with support at the origin. This singularity is related, through Einstein field equations, to a point source. Coordinate invariance and independence on the choice of differentiable structure of the results are addressed.

I. INTRODUCTION

The $(2+1)$-dimensional black hole of Bañados, Teitelboim and Zanelli (BTZ) [1,2] provides us with a useful model to study various classical and quantum aspects of black hole physics [3]. The BTZ black hole shares many of the properties of the more complicated $3+1$-dimensional Kerr black hole. However, it differs from the Kerr solution in which it is asymptotically anti-de Sitter rather than asymptotically flat. Furthermore, by admitting closed time-like curves, the rotating BTZ black hole has no curvature singularities. Nevertheless, when there is no angular momentum the spacetime fails to be Hausdorff at the origin and turns out to be singular [2].

The purpose of this work is to analyse the distributional BTZ black hole geometry for the non-rotating case. There are several reasons, both mathematical and physical, to carry out such analysis. The non-rotating BTZ black hole provides an example of a singular spacetime whose singularities can not be identified with the unboundedness of some scalar constructed from the curvature tensor. In this sense, it resembles the well known conical singularities [4] whose meaning has attracted wide interest for many years [5]. On the other hand, regularization procedures required to multiply distributions need not be invoked in the calculation of the distributional non-rotating BTZ black hole curvature tensor. We show that the non-rotating BTZ black hole metric belongs to the class of semi-regular metrics, as defined in Ref. [6], for which the curvature tensor field has direct distributional meaning. Note that not many semi-regular metrics are known. It was shown that the $(3+1)$-dimensional Minkowski metric with an angular deficit, and a certain kind of traveling wave metric are semi-regular [6]. Recently, the metric associated to the $(2+1)$-dimensional spacetime around a point source [4] was demonstrated to be a third example [7].

The distributional description of the BTZ black hole spacetime is carried out in Schwarzschild coordinates and in Kerr-Schild coordinates. We find the distributional Ricci and Einstein tensor fields for the non-rotating BTZ black hole, which turn out to be equivalent in both coordinate systems. This indicates that, although the intermediate calculations depend upon the choice of coordinates, the final results does not. Remarkably, a complete agreement with what is physically expected is found. The distributional curvature tensor becomes, besides the constant curvature part, a δ-distribution supported at the origin. Furthermore, this singularity is related through Einstein field equations to a point source.

The paper is organized as follows. In the next section, following the procedure of [6], the distributional curvature and Einstein tensor fields are found for the non-rotating BTZ solution in Schwarzschild coordinates. In section III, we carry out the distributional analysis using the Kerr-Schild form of the BTZ solution. The last section is devoted to summarize and discuss the coordinate invariance (and differentiable structure dependence) of our results.

II. BTZ BLACK HOLE IN SCHWARZSCHILD COORDINATES

The non-rotating BTZ black hole solution [1,2] written in Schwarzschild coordinates is given by

$$g_{ab} = -(-m + \frac{r^2}{l^2})dt_a dt_b + (-m + \frac{r^2}{l^2})^{-1} dr_a dr_b + r^2 d\varphi_a d\varphi_b, \tag{1}$$

where $-\infty < t < \infty$, $0 < r < \infty$ and $0 \leq \varphi < 2\pi$, with the surfaces $\varphi = 0, 2\pi$ identified. The dimensionless quantity m is the mass parameter. For this metric we have

$$R_{abc}{}^{d} = g_{ac}R_{b}{}^{d} - g_{bc}R_{a}{}^{d} + \delta_{b}{}^{d}R_{ac} - \delta_{a}{}^{d}R_{bc} - \frac{1}{2}(g_{ac}\delta_{b}{}^{d} - g_{bc}\delta_{a}{}^{d})R, \tag{2}$$

where

$$R_{ab} = -\frac{2}{l^2} g_{ab}. \tag{3}$$

Hence, (1) has constant negative curvature.

The BTZ metric (1) is a solution of the vacuum Einstein field equations with cosmological constant $\Lambda = -1/l^2$,

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 0 \tag{4}$$

and may be obtained by identifying certain points of (the covering manifold of) the anti-de Sitter space [1,2]. For $m > 0$, (1) describes a black hole of mass m with horizon at $r_+ = \sqrt{m}l$. Note that the black hole is characterized, in the present context, by the existence of an event horizon and not by the existence of a region of large curvature. The solution with $-1 < m < 0$ may be associated to the metric generated by a point source at the origin [8]. The solution with $m = -1$ is anti-de Sitter space. The massless black hole, $m = 0$, is commonly considered as the vacuum state. For a review of the properties of BTZ black holes see [3].

In (1), $\sqrt{g_{\varphi\varphi}} = r$ represents the radius associated with the proper circumference. Therefore with $x = r\cos\varphi$ and $y = r\sin\varphi$ we have

$$g_{ab} = \eta_{ab} + (1 + m - \frac{r^2}{l^2}) dt_a dt_b - \left(\frac{l^2 + ml^2 - r^2}{ml^2 - r^2}\right) \frac{1}{r^2} (x dx_a + y dy_a)(x dx_b + y dy_b), \tag{5}$$

where $r = \sqrt{x^2 + y^2}$ and η_{ab} is the ordinary Minkowski metric on $\mathcal{R}^3$. The metric (5) is singular when $r = 0$, except for $m = -1$. This singularity is not a coordinate singularity. The spinless BTZ black hole is geodesically incomplete [9] and the singularity at $r = 0$ corresponds to fixed points of the identifications from which the BTZ solution is obtained [10]. Assuming that (5) can be extended to $r = 0$, we look for the distributional curvature tensor and its relation with a possible distributional source.

Suppose $(\mathcal{M}, g_{ab})$ are given such that

1. g_{ab} and $(g^{-1})^{ab}$ exist almost everywhere and are locally integrable,
2. the weak first derivative $\nabla_c g_{ab}$ of g_{ab} in a smooth derivative operator ∇_c exists and the tensors

$$C^c_{ab} \equiv \frac{1}{2} (g^{-1})^{cd} (\nabla_a g_{bd} + \nabla_b g_{ad} - \nabla_d g_{ab}), \tag{6}$$

   and $C^d_{m[b} C^m_{a]c}$ are locally integrable.

These are the minimal conditions for ${R_{abc}}^d$ to be definable as a distribution by the usual coordinate formula,

$${R_{abc}}^d = {\tilde{R}_{abc}}{}^d + 2\nabla_{[b} C^d_{a]c} + 2 C^d_{m[b} C^m_{a]c}, \tag{7}$$

where ${\tilde{R}_{abc}}{}^d$ is the curvature tensor associated to the smooth derivative operator ∇_c [6]. We shall say that g_{ab} is a semi-regular metric.

A semi-regular metric may have no distributional Einstein tensor due to the fact that contractions of the metric with the curvature tensor may have no sense as distributions. Stronger conditions can be imposed to isolate the class of metrics for which the distributional meaning of the Einstein tensor is ensured, but then the distributional curvature tensor must have its support on a submanifold of codimension of at most one [11]. Metrics for surface layers [12] lie in this class, but neither strings nor point particles can be described by metrics in this class. Alternatively, by considering Colombeau's generalized functions [13], distributional curvatures can be defined for those cases where a direct calculation would not work. This approach has been used to obtain the distributional curvature associated with a conical singularity [14–16] and to the analysis of the distributional Schwarzschild geometry [17], but we will not consider it here.

We shall prove that (5) is a semi-regular metric. We take for the differentiable structure that in which t, x and y form a smooth chart. For a test tensor field U^{ab} on $\mathcal{R}^3$ we have

$$g_{ab}[U^{ab}] \equiv \int_{\mathcal{R}^3} g_{ab} U^{ab} \omega_\eta = \int_{\mathcal{R}^3} \eta_{ab} U^{ab} \omega_\eta + \int_{\mathcal{R}^3} (1 + m - \frac{r^2}{l^2}) U^{tt} \omega_\eta - \int_{\mathcal{R}^3} \left(\frac{l^2 + ml^2 - r^2}{ml^2 - r^2}\right) dr_a dr_b U^{ab} \omega_\eta, \tag{8}$$

where ω_η is the volume element associated to η_{ab} and it is understood that all tensor components are Cartesian components as functions of Cartesian coordinates. For $m > 0$ we require that U^{ab} be a test tensor with support on $r < \sqrt{m}l$, while for $m < 0$ we simply require that U^{ab} be a test tensor of compact support. The black vacuum can not be handled in the Schwarzschild-type coordinates because the last term in the right-hand side of (8) is not locally integrable for $m = 0$. Note that, with this choice, the natural volume element ω_g associated to g_{ab} agrees with the volume element ω_η of η_{ab}.

It follows that

$$g_{ab}[U^{ab}] = \int_{\mathcal{R}^3} \eta_{ab}U^{ab}\omega_\eta + \int_{\mathcal{R}^3}(1+m-\frac{r^2}{l^2})U^{tt}\omega_\eta \\ -\int_{\mathcal{R}^3}\left(\frac{l^2+ml^2-r^2}{ml^2-r^2}\right)(\cos^2\varphi U^{xx}+\cos\varphi\sin\varphi(U^{xy}+U^{yx})+\sin^2\varphi U^{yy})\omega_\eta. \tag{9}$$

Therefore, g_{ab} is locally integrable for $m \neq 0$.

Next, let U_{ab} be a test tensor field on $\mathcal{R}^3$. For

$$(g^{-1})^{ab} \equiv \eta^{ab} + \left(\frac{l^2+ml^2-r^2}{ml^2-r^2}\right)\partial_t^a\partial_t^b - (1+m-\frac{r^2}{l^2})\partial_r^a\partial_r^b, \tag{10}$$

we have

$$(g^{-1})^{ab}[U_{ab}] = \int_{\mathcal{R}^3}\eta^{ab}U_{ab}\omega_\eta + \int_{\mathcal{R}^3}\left(\frac{l^2+ml^2-r^2}{ml^2-r^2}\right)\ U_{tt}\ \omega_\eta \\ -\int_{\mathcal{R}^3}\left(1+m-\frac{r^2}{l^2}\right)(\cos^2\varphi U_{xx}+\cos\varphi\sin\varphi(U_{xy}+U_{yx})+\sin^2\varphi U_{yy})\ \omega_\eta. \tag{11}$$

Therefore, $(g^{-1})^{ab}$ is locally integrable for $m \neq 0$.

We now calculate the weak derivative in η_{ab} of g_{ab}. Let U^{abc} be a test tensor field. We find

$$\nabla_c g_{ab}[U^{cab}] \equiv -\int_{\mathcal{R}^3} g_{ab}\nabla_c U^{cab}\omega_\eta = \int_{\mathcal{R}^3} W_{cab}U^{cab}\omega_\eta, \tag{12}$$

where W_{cab} is the locally integrable but not locally square integrable tensor given by

$$W_{cab} = -\frac{2r}{l^2}dr_c(dt_a dt_b + (m-\frac{r^2}{l^2})^{-2}dr_a dr_b) - \frac{1}{r}\left(\frac{l^2+ml^2-r^2}{ml^2-r^2}\right)rd\varphi_c(rd\varphi_a dr_b + rdr_a d\varphi_b). \tag{13}$$

From (6), we find

$$C^c_{ab} = \frac{-r(ml^2-r^2)}{l^4}dt_a dt_b\partial_r^c - \frac{r}{ml^2-r^2}(dr_a dt_b + dt_a dr_b)\partial_t^c \\ +\frac{r}{ml^2-r^2}dr_a dr_b\partial_r^c + \frac{1}{r}\frac{ml^2+l^2-r^2}{l^2}r^2 d\varphi_a d\varphi_b\partial_r^c, \tag{14}$$

which is locally integrable. On the other hand,

$$2C^d_{m[b}C^m_{a]c} = \frac{2r}{ml^2-r^2}(dt_a dr_b - dr_a dt_b)\left(\frac{r}{ml^2-r^2}dr_c\partial_t^d - \frac{r(ml^2-r^2)}{l^4}dt_c\partial_r^d\right), \tag{15}$$

which is locally integrable. Hence, g_{ab} is a semi-regular metric in the differentiable structure chosen.

Now, contracting (7) and using (14,15) we find for the Ricci tensor of g_{ab}

$$R_{ac}[U^{ac}] = -\int_{\mathcal{R}^3} C^b_{ac}\nabla_b U^{ac}\omega_\eta - \int_{\mathcal{R}^3} C^b_{ma}C^m_{bc}U^{ac}\omega_\eta, \tag{16}$$

where

$$-\int_{\mathcal{R}^3} C^b_{ac}\nabla_b U^{ac}\omega_\eta = -\int_{r=\varepsilon} dr_b C^b_{ac}U^{ac}\sigma + \int_{r>\varepsilon}\nabla_b C^b_{ac}U^{ac}\omega_\eta, \tag{17}$$

with

$$\nabla_b C^b_{ac} = -\frac{2(ml^2-2r^2)}{l^4} dt_a dt_c + 2\frac{ml^2}{(ml^2-r^2)^2} dr_a dr_c - 2\frac{r^2}{l^2} d\varphi_a d\varphi_c, \tag{18}$$

which is a locally integrable tensor and where it is understood that $\varepsilon \to 0$.

From (16) and (15,17,18) we find

$$R_{ac}[U^{ac}] = \pi(1+m)\int dt \left(U^{xx}(t,\vec{0}) + U^{yy}(t,\vec{0})\right) - \frac{2}{l^2}\int_{\mathcal{R}^3} g_{ac} U^{ac} \omega_\eta, \tag{19}$$

where g_{ac} is the locally integrable tensor defined by (9). Thus we obtain

$$R_{ac} = \pi(1+m)\delta^{(2)}_{(0)}(dx_a dx_c + dy_a dy_c) - \frac{2}{l^2} g_{ac}. \tag{20}$$

Note that $\forall m > -1$, the Ricci tensor has a singular part proportional to a δ distribution. As expected, for $m = -1$ the singular part of the curvature is absent, as follows from the fact that in this case we have AdS_3 spacetime. Since for $-1 < m < 0$, there is no horizon, we have a naked singularity at $r = 0$. For $m > 0$ we have a singularity at $r = 0$ hidden by a horizon at $r_+ = \sqrt{m}l$. As stated before, for $m = 0$ the metric (8) is not a semi-regular metric and the massless BTZ black hole can not be properly discussed from the above derivation.

We now calculate the Einstein tensor of g_{ab}. Define

$$G^a_b = R^a_b - \frac{1}{2}(g^{-1})^{cd}\tilde{R}_{cd}\delta^a_b + (g^{-1})^{cd} C^e_{m[c} C^m_{e]d}\delta^a_b + \nabla_{[c}\left(C^e_{e]d}(g^{-1})^{cd}\right)\delta^a_b + C^e_{d[c}\nabla_{e]}(g^{-1})^{cd}\delta^a_b, \tag{21}$$

where

$$R^a_b = (g^{-1})^{ac}\tilde{R}_{cb} + 2\nabla_{[c}\left(C^c_{d]b}(g^{-1})^{ad}\right) + 2C^c_{b[c}\nabla_{d]}(g^{-1})^{ad} + 2(g^{-1})^{ad} C^c_{m[c} C^m_{d]b}. \tag{22}$$

We shall say that (21) is the Einstein tensor distribution of (5), whenever each term in the right-hand sides of (21,22) defines a distribution.

From (14,15) and (22) we find

$$\begin{aligned} R^d_c &= \nabla_b\left((g^{-1})^{ad} C^b_{ac}\right) \\ &= \nabla_b\left(-\frac{r}{l^2}\partial^d_t dt_c \partial^b_r - \frac{rl^2}{(ml^2-r^2)^2}\partial_t dr_c \partial^b_t + \frac{r}{l^2}\partial^d_r dt_c \partial^b_t - -\frac{r}{l^2}\partial^d_r dr_c \partial^b_r + \frac{ml^2+l^2-r^2}{rl^2}\partial^d_\varphi d\varphi_c \partial^b_r\right) \end{aligned} \tag{23}$$

Note that the right-hand side of (23) is the derivative of a locally integrable tensor. Therefore (23) defines a distribution.

An analogous calculations to that of (20) leads to

$$R^a_b = \pi(1+m)\delta^{(2)}_{(0)}(\partial^a_x dx_b + \partial^a_y dy_b) - \frac{2}{l^2}(\partial^a_t t_b + \partial^a_x dx_a + \partial^a_y dy_b) \tag{24}$$

Finally, from (14,15) and (21,24) we obtain

$$G^a{}_b - \frac{1}{l^2}(\partial^a_t t_b + \partial^a_x dx_a + \partial^a_y dy_b) = -\pi(1+m)\delta^{(2)}_{(0)}\partial^a_t dt_b. \tag{25}$$

Remarkably enough, the right-hand side of (25) resembles the physically expected result for the distributional energy momentum tensor $T^a{}_b = -m\delta^{(3)}_{(0)}\partial^a_t dt_b$ of the Schwarzschild four dimensional black hole [18–20,7,17].[1]

Now, let us consider the dependence of these results on the coordinate system. Note that whether or not a metric is semi-regular depends in general on the differentiable structure imposed on the manifold. In this section, the choice of the manifold differentiable structure was made on the basis of an interpretation of the coordinate system in which the metric is given: we use Cartesian coordinates associated with the Schwarzschild coordinates. In the next section the distributional curvature and Einstein tensor fields are evaluated using the Kerr-Schild form of the BTZ metric. This amounts to change both the coordinates and the differentiable structure.

[1] The Schwarzschild metric is not a semi-regular metric and cannot be handled with the methods used here to obtain its distributional curvature [7].

III. BTZ BLACK HOLE IN KERR-SCHILD COORDINATES

In previous works the Kerr-Schild form of the Schwarzschild metric has been proved to be useful, from both conceptual and technical points of view, for the analysis of the distributional Schwarzschild geometry from quite different approaches [18,20,17]. In the following we shall prove that the non-rotating BTZ solution in Kerr-Schild coordinates is a semi-regular metric.

The AdS_3 black hole solution of BTZ is given in the Kerr-Schild form by [21]

$$g_{ab} = \eta_{ab} + (1 + m - \frac{r^2}{l^2})k_a k_b, \tag{26}$$

where $r = \sqrt{x^2 + y^2}$ and

$$k_a = dt_a + \frac{1}{r}(xdx_a + ydy_a), \tag{27}$$

with $k^a = \eta^{ab}k_b$ a null vector field with respect to η_{ab} and g_{ab}. It follows that there are two metrical structures, η_{ab} and g_{ab}, associated to the manifold. We choose as the underlying manifold structure that of $\mathcal{R}^3$ with the smooth metric η_{ab} in Cartesian coordinates $\{t, x, y\}$. Note that in Kerr-Schild coordinates r is a spacelike coordinate $\forall r > 0$, which is not the case in Schwarzschild type coordinates. Note also that the natural volume element ω_g associated to g_{ab} agrees with the volume element ω_η of η_{ab}.

Now,

$$(g^{-1})^{ab} = \eta^{ab} - (1 + m - \frac{r^2}{l^2})\left(\partial_t^a - \frac{1}{r}(x\partial_x^a + y\partial_y^a)\right)\left(\partial_t^b - \frac{1}{r}(x\partial_x^b + y\partial_y^b)\right). \tag{28}$$

Clearly, g_{ab} and $(g^{-1})^{ab}$ are locally integrable $\forall m$ (Actually, g_{ab} and $(g^{-1})^{ab}$ are locally bounded).

The weak derivative in η_{ab} of g_{ab} exists almost everywhere and is given by

$$\nabla_c g_{ab}[U^{cab}] = \int_{\mathcal{R}^3} W_{cab}U^{cab}\omega_\eta, \tag{29}$$

where

$$W_{cab} = -\frac{r}{l^2}dr_c(dt_a + dr_a)(dt_b + dr_b) + \frac{1}{r}(1 + m - \frac{r^2}{l^2})rd\varphi_c\left(rd\varphi_a(dt_b + dr_b) + (dt_a + dr_a)rd\varphi_b\right), \tag{30}$$

which is locally integrable $\forall m$.

From (6), it follows

$$\begin{aligned} C^c_{ab} = {} & \frac{2r}{l^2}\left(dr_a(dt_b + dr_b) + dr_b(dt_a + dr_a)\right)(\partial_t^c - \partial_r^c) + \frac{2r}{l^2}(1 + m - \frac{r^2}{l^2})(dt_a + dr_a)(dt_b + dr_b)(\partial_t^c - \partial_r^c) \\ & + \frac{2r}{l^2}(dt_a + dr_a)(dt_b + dr_b)\partial_r^c - \frac{2}{r}(1 + m - \frac{r^2}{l^2})r^2 d\varphi_a d\varphi_b(\partial_t^c - \partial_r^c), \end{aligned} \tag{31}$$

which is locally integrable $\forall m$. Note that $C^b_{ab} = 0$.

Finally, from (31) we find

$$\begin{aligned} 2C^d_{m[b}C^m_{a]c} = {} & 2\frac{r^2}{l^4}(dt_a dr_b - dr_a dt_b)(dt_c + dr_c)(\partial_t^d - \partial_r^d) \\ & -\frac{1}{l^2}(1 + m - \frac{r^2}{l^2})\left((dt_a + dr_a)rd\varphi_b - rd\varphi_a(dt_b + dr_b)\right)rd\varphi_c(\partial_t^d - \partial_r^d), \end{aligned} \tag{32}$$

which is locally integrable $\forall m$. Therefore the metric (26) is semi-regular. Furthermore, since (26) is a semi-regular metric $\forall m$, we can now consider the distributional geometry of the BTZ black hole including the $m = 0$ black vacuum.

We now calculate the Ricci tensor of (26). We find

$$R_{ac}[U^{ac}] = -\int_{\mathcal{R}^3} C^b_{ac}\nabla_b U^{ac}\omega_\eta - \int_{\mathcal{R}^3} \frac{2r}{l^2}(dt_a + dr_a)(dt_c + dr_c)U^{ac}\omega_\eta. \tag{33}$$

An analogous calculation to that of (20) leads to

$$R_{ac} = \pi(1+m)\delta^{(2)}_{(0)}(dx_a dx_c + dy_a dy_c) - \frac{2}{l^2} g_{ac}, \tag{34}$$

where g_{ac} is the locally integrable tensor (26). Note that (34) is equivalent to (20) which was obtained using Schwarzschild type coordinates.

Calculations analogous to the ones done previously show that

$$R^a{}_b = \pi(1+m)\delta^{(2)}_{(0)}(\partial^a_x dx_b + \partial^a_y dy_b) - \frac{2}{l^2}(\partial^a_t t_b + \partial^a_x dx_a + \partial^a_y dy_b) \tag{35}$$

and

$$G^a{}_b - \frac{1}{l^2}(\partial^a_t t_b + \partial^a_x dx_a + \partial^a_y dy_b) = -\pi(1+m)\delta^{(2)}_{(0)}\partial^a_t dt_b. \tag{36}$$

IV. DISCUSSIONS

As noted above, the distributional Ricci tensor fields (20) and (34) are equivalent, as is the case with the mixed index versions (24) and (35). We take (34) and (35), which are valid $\forall m$, as the distributional R_{ab} and $R^a{}_b$ Ricci tensor fields. The non-rotating BTZ black hole geometry is singular and this singularity is a curvature singularity proportional to a δ-distribution supported at the origin. As follows from (34) and (35), even for the $m=0$ black vacuum this singularity is present. As expected, for $m=-1$ (anti-de Sitter spacetime) a non-singular spacetime is recovered.

On the other hand, (25) and (36) are equivalent. Hence, we take (36) as the distributional Einstein tensor field $G^a{}_b$ of the non-rotating BTZ black hole. As expected on physical grounds, the distributional energy momentum tensor T^a_b of the BTZ black hole geometry is then given by

$$T^a_b = -(1+m)\delta^{(2)}_{(0)}\partial^a_t dt_b, \tag{37}$$

where we have set the gravitational constant G equal to $\frac{1}{8}$. Note that (37) is a non-zero distribution for $m=0$. The constant shift in the mass is due to the fact that in (1) the zero point of energy has been set so that the mass vanishes when the horizon size, the length of the minimal geodesic of the horizon, goes to zero [2]. As follows from the fact that the singular parts of the distributional curvature and Einstein tensor fields are equal to zero only for $m=-1$, if the zero of energy is adjusted so that anti-de Sitter space has zero mass then $1+m \to m$.

Finally, let us briefly discuss the coordinate independence of these results. As already mentioned in section II, whether or not a given metric is semi-regular depends in general on the differentiable structure imposed on the underlying manifold. In sections II and III, the choice of the differentiable structure was made on the basis of an interpretation of the coordinate system in which the metric is given. In this sense, the choice of coordinates determines the differentiable structure of the underlying manifold. Here we find that the distributional curvature and Einstein tensor fields are well defined and that they are equivalent in both descriptions. Thus, in a restricted sense, the present distributional treatment in Schwarzschild and Kerr-Schild coordinates provides invariant results for the distributional curvature and Einstein tensor fields of the BTZ black hole geometry.

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[1] M. Bañados, C. Teitelboim and J. Zanelli, *Phys. Rev. Lett.* **69** (1992) 1849, hep-th/9204099.
[2] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Phys. Rev.* **D48** (1993) 1506, gr-qc/9302012.

[3] S. Carlip, *Class. Quantum Grav.* **12** (1995) 2853, gr-qc/9506079.
[4] S. Deser, R. Jackiw and G. 't Hooft, *Ann. Phys. (N.Y.)* **152** (1984) 220.
[5] G. 't Hooft, *Class. Quantum Grav.* **10** (1993) S79.
[6] D. Garfinkle, *Class. Quant. Grav.* **16** (1999) 4101, gr-qc/9906053.
[7] N. R. Pantoja and H. Rago, *Int. J. Mod. Phys.* **D** (to be published), gr-qc/0009053.
[8] S. Deser and R. Jackiw, *Ann. Phys. (N.Y.)* **153** (1984) 405.
[9] N. Cruz, C. Martinez and L. Peña, *Class. Quantum Grav.* **11** (1994) 2731, gr-qc/9401025.
[10] A. R. Steif, *Phys. Rev. D* **53** (1996) 5521, hep-th/9504012.
[11] R. P. Geroch and J. Traschen, *Phys. Rev. D* **36** (1987) 1017.
[12] W. Israel, *Nuovo Cimento* **B44** (1966) 1. *Erratum:* **B48** (1967) 463.
[13] J. F. Colombeau, *New Generalized Functions and Multiplication of Distributions* (North-Holland, 1984).
[14] C. J. S. Clarke, J. A. Vickers and J. P. Wilson *Class. Quantum Grav.* **13** (1996) 2485, gr-qc/9605060.
[15] J. P. Wilson *Class. Quantum Grav.* **14** (1997) 3337, gr-qc/9705032.
[16] J. A. Vickers and J. P. Wilson *Class. Quantum Grav.* **16** (1999) 579, gr-qc/9806068.
[17] J. M. Heinzle and R. Steinbauer, *J. Math. Phys.* **43** (2002) 1493, gr-qc/0112047.
[18] P. E. Parker, *J. Math. Phys.* **20** (1979) 1423.
[19] H. Balasin and H. Nachbagauer, *Class. Quantum Grav.* **10** (1993) 2271, gr-qc/9305009.
[20] H. Balasin and H. Nachbagauer, *Class. Quantum Grav.* **11** (1994) 1453, gr-qc/9312028.
[21] H. Kim, *Phys. Rev.* **D59** (1999) 064002-1, gr-qc/9809047.