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UNIFORM ORDERED SPECTRAL DECOMPOSITIONS

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RESUMEN

Introducimos la noción de UOSD-multiplicidad de una proyección P relativa a una medida espectral E(.) con la CGS-propiedad y la comparamos con la noción de multiplicidad introducida por Halmos [2]. También se dan varias caracterizaciones para que una medida espectral tanga la CGS-propiedad.

ABSTRACT

We introduce the notion of UOSD-multiplicy of a projection P relative to a spectral measure E(.) with the CGS-property and compare it with the notion of multiplicy introduced by Halmos [2]. Also are given some characterizations for a spectral measure to have the CGS-property.

In our earlier work [4] we introduced the notion of ordered spectral decomposition (OSD, in abbreviation) of a Hilbert space relative to a spectral measure E(.) and defi-

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ned the OSD-multiplicity of a projection P commuting with $E(.)$. Here we introduce the concepts of uniform OSD and UOSD-multiplicity and compare the concept of multiplicity in Halmos [2] with the OSD and UOSD-multiplicities. Also we obtain various characterizations for a spectral measure to have the CGS-property.

1. PRELIMINARIES. In this section we fix the terminology and notations and give some definitions and results from the literature which are needed in the sequel.

$\mathcal{S}$ denotes a σ-algebra of subsets of a set $X(\neq\phi)$. H is a (complex) Hilbert space and $E(.)$ is a spectral measure on $\mathcal{S}$ with values in projections of H. The closed subspace generated by a subset $\mathfrak{X}$ of H is denoted by $[\mathfrak{X}]$. For a vector $x \in H$, $Z(x) = [E(\sigma)x: \sigma \in \mathcal{S}]$. $\sum_{i\in J} \oplus M_i$ denotes the orthogonal direct sum of the subspaces M_i of some Hilbert space.

W is the Von Neumann algebra generated by the range of $E(.)$ and W' is the commutant of W. If $W' = \Sigma \oplus W'Q_n$ is the type I_n direct sum decomposition of W', then the central projections $Q_n (\neq 0)$ are unique (such that $W'Q_n$ is of type I_n) and in the sequel Q_n will denote these central projections. For $x \in H$, $[Wx] = [Ax: A \in W]$ and, sometimes, also denotes the orthogonal projection with the range $[Wx]$. For a projection $P' \in W'$, $C_{P'}$ denotes the central support of P'. Other terminology in Von Neumann algebras is standard and we follow Dixmier [1].

As was observed in [5] a projection P' in W' is abelian if and only if P' is a row projection in the sense of [2] and the column $C(P')$ generated by P' as in [2] is the same as $C_{P'}$.

NOTATION 1.1. Let P be a projection in W. The multiplicity (respy. uniform multiplicity) of P in the sense of Halmos [2] will be referred to as its H-multiplicity (respy. UH-multiplicity) relative to $E(.)$.

As was noted in [5] Theorem 64.4 of Halmos [2] can be interpreted as follows:

THEOREM 1.2. A non-zero projection F in W has UH-multiplicity n if and only if there exists an orthogonal family $\{E'_\alpha\}_{\alpha\varepsilon J}$ of abelian projections in W' such that card. $J = n$, $C_{E'_\alpha} = F$ and $\sum_{\alpha\varepsilon J} E'_\alpha = F$. In other words, F has UH-multiplicity n if and only if $W'F$ is of type I_n.

Consequently, the following proposition is immediate.

PROPOSITION 1.3. A non-zero projection P in W has UH-multiplicity n if and only if $P \leq Q_n$.

DEFINITION 1.4. $E(.)$ is said to have the CGS-property (i.e. countable generating set property) in H if there exists a countable set $\mathfrak{X}$ in H such that $[E(\sigma)x: \sigma \varepsilon \mathbf{S}, x\varepsilon\mathfrak{X}]=H$.

Let $\rho(x) = \|E(.)x\|^2$. Then $\rho(x)$ is a finite measure on $\mathcal{S}$. We say that $\rho(x_2)$ is absolutely continuous with respect to $\rho(x_1)$ and write $\rho(x_2) \ll \rho(x_1)$ (or $\rho(x_1) \gg \rho(x_2)$) if $\rho(x_1)(\sigma) = 0$ implies $\rho(x_2)(\sigma) = 0$.

DEFINITION 1.5. Let $\{x_i\}_1^N$, $N \in \mathbb{N} \cup \{\infty\}$, be a countable set of non-zero vectors in H such that (i) $H = \sum_1^N \oplus Z(x_i)$ and (ii) $\rho(x_1) \gg \rho(x_2) \gg \ldots$ Then we say that $H = \sum_1^N \oplus Z(x_i)$ is an OSD of H relative to E(.).

The cardinal number $N \in \mathbb{N} \cup \aleph_0$ in the above definition is uniquely fixed by E(.) and is called the OSD-multiplicity of E(.). If P is a projection commuting with E(.) and PE(.) has the CGS-property in H, then the OSD-multiplicity of PE(.) is called the OSD-multiplicity of P. Besides, it has been shown in [4] that E(.) has the CGS-property in H if and only if H has an OSD relative to E(.).

2. UOSD-MULTIPLICITY OF PROJECTIONS.

We introduce the concepts of UOSDs and UOSD-multiplicity relative to a spectral measure E(.) with the CGS-property in H and show that for a projection P in W the UOSD-multiplicity and the UH-multiplicity are one and the same when P is countably decomposable in W.

DEFINICION 2.1. An OSD $H = \sum_1^N \oplus Z(x_i)$ relative to E(.)

is said to be a uniform OSD(UOSD, in abbreviation) of H if $\rho(x_1) \equiv \rho(x_2) \equiv \ldots$, where $\mu \equiv \nu$ if $\mu \ll \nu$ and $\nu \ll \mu$.

The following proposition is immediate from Theorem 1.(ii) of [4].

RROPOSITION 2.2. If H has a UOSD relative to E(.), then all the OSDs of H relative to E(.) are UOSDs.

DEFINITION 2.3. If H has a UOSD relative to E(.) then the UOSD-multiplicity of E(.) is defined to be the same as its OSD-multiplicity. If P is a projection of H commuting with E(.) and if P E(.) has UOSD-multiplicity n, then we say that P has UOSD-multiplicity n relative to E(.).

The following simple example shows that, in general, the OSD-multiplicity and H-multiplicity of a projection P relative to E(.) are not the same even though H is finite dimensional.

EXAMPLE 2.4. Let $H = \mathbb{C}^5$, $\mathcal{S} = \{\phi, \{\lambda_1\}, \{\lambda_2\}, \{\lambda_1, \lambda_2\}\}$, $\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_1 \neq \lambda_2$ and E(.) be a spectral measure on $\mathcal{S}$ given by $E(\{\lambda_1\})H = [e_1, e_2]$ and $E(\{\lambda_2\})H = [e_3, e_4, e_5]$, where $e_1 = (1,0,0,0,0)$, $e_2 = (0,1,0,0,0)$, etc. Since any maximal orthogonal family of row projections (in the sense of Halmos [2]) $\{E'_\alpha\}$ in W' with $C_{E'_\alpha} = I$ consists of just two members, the H-multiplicity of I is 2. On the other hand, if $x_1 = e_1 + e_3$, $x_2 = e_2 + e_4$ and $x_3 = e_5$, then $H = \sum_1^3 \oplus Z(x_i)$ is an OSD of H since $\rho(x_1) \equiv \rho(x_2) >> \rho(x_3)$. Thus the OSD-multiplicity of I is 3.

The following result is well-known in the theory of Von Neumann algebras, whose proof is indicated also on p.108 of [2]. Using this result we compare the UH-multiplicity and UOSD-multiplicity of a projection.

LEMMA 2.5. Let P' be an abelian projection in W'. If the central support $C_{P'}$ of P' is countably decomposable in W, then P' is cyclic.

THEOREM 2.6. Let P be a countably decomposable non-zero projection in W'. Then P has UH-multiplicity $N \leq \aleph_0$ if and only if P has UOSD-multiplicity N (relative to E(.)).

PROOF. Suppose the UH-multiplicity of P is $N \leq \aleph_0$. Then by Theorem 1.2 there exists an orthogonal family $\{P_j'\}_{j\epsilon J}$ of abelian projections in W' such that card. J=N, $C_{P_j'} = P$ and $P = \sum_{j \epsilon J} P_j'$. Let $J = \{1,2,\ldots,N\}$. By Lemma 2.5 there exists $x_j \epsilon P_j' H$ such that $P_j' = [Wx_j]$, $j \epsilon J$. Thus $PH = \sum_{j\epsilon J} \oplus [Wx_j] = \sum_{j=1}^{N} \oplus Z(x_j)$. Besides, by Theorem 66.2 of [2], $C(\rho(x_j)) = C_{[Wx_j]} = C_{P_j'} = P$ for all j. Therefore, by Theorem 65.2 of [2], $\rho(x_j) \equiv \rho(x_{j'})$ for $j, j' \in J$. Hence the condition is necessary.

Conversely, if P has UOSD-multiplicity N, then clearly $N \leq \aleph_0$. Let $PH = \sum_1^N \oplus Z(x_i)$ be an OSD of PH relative to $PE(.)$. Then by Proposition 2.2, $\rho(x_1) \equiv \rho(x_2) \equiv \ldots$ Consequently, by Theorem 66.2 of [2] we conclude that $C_{[Wx_1]} = C_{[Wx_2]} = \ldots = Q$(say). Clearly, $P = \sum_1^N [Wx_i] \leq Q$. As $P \in W$, $[Wx_i] \leq C_{[Wx_i]} = P$ so that $Q = P$. Since each $[Wx_i]$ is an abelian projection in W' by Theorem 60.2 of [2], from Theorem 1.2 it follows that P has UH-multiplicity N.

3. SOME CHARACTERIZATIONS OF THE CGS-PROPERTY.

In terms of the existence of OSDs and OSRs of H the CGS-property of a spectral measure $E(.)$ is characterized in [4]. The following Theorem gives some more characterizations of this property.

THEOREM 3.1. Let $E(.)$ be a spectral measure on $\mathcal{S}$ with values in projections of H. Then the following statements are equivalent.

(i) Every projection of UH-multiplicity N in W is countably decomposable in W and $n \leq \aleph_0$.

(ii) The projections Q_n are countably decomposable in W and $Q_n = 0$ for $n > \aleph_0$.

(iii) Every projection in W is countably decomposable in W and has H-multiplicity $n \leq \aleph_0$.

(iv) Every projection of UH-multiplicity in W is countably decomposable in W'.

(v) The projections Q_n are countably decomposable in W'.

(vi) Every projection in W is countably decomposable in W'.

(vii) Every non-zero projection of UH-multiplicity in W has UOSD-multiplicity (and hence they are equal).

(viii) E(.) has the CGS-property in H.

PROOF.

(i) =>(ii) Let $Q_n \neq 0$. Then by Proposition 1.3, Q_n has UH-multiplicity n. Therefore, (ii) is immediate from (i).

(ii)=>(iii) If P is a non-zero projection in W, then by (ii) $P = \sum_{n \leq \aleph_0} Q_n P$. Being Q_n countably decomposable in W, it follows that the same is true for P. Then the H-multiplicity of P $= \min \{n: PQ_n \neq 0\} \leq \aleph_0$ by Theorem 64.2 of [2] and by Proposition 1.3.

(iii) =>(iv) Let P be a non-zero projection of UH-multiplicity n. By (iii), $n \leq \aleph_0$. By Theorem 1.2 there exists an orthogonal family $\{E_1'\}_1^n$ of abelian projections in W' such that $P = \sum_1^n E_i'$ and $C_{E_i'} = P$ for all i. Now by (iii) and Lemma 2.5 there exist vectors x_i in PH such that $[Wx_i] = E_i'$. If $\mathfrak{X} = \{x_i\}_1^n$, then clearly $PH = [W\mathfrak{X}]$ so that by Lemma 3.3.9 of [3] P is countably decomposable in W'.

(iv)$\Longrightarrow$(v) This is immediate, since Q_n has UH multiplicity n by Proposition 1.3.

(v)$\Longrightarrow$(vi) Let P be a non-zero projection in W. Then $P = \sum_{n \leq \dim H} P Q_n$ and by (v) Q_n are countably decomposable in W'. To prove that P is countably decomposable in W', it suffices to show that $Q_n = 0$ for $n > \aleph_0$. If $Q_n \neq 0$, as Q_n has UH-multiplicity n by Proposition 1.3, there exists an orthogonal family $\{E_\alpha'\}_{\alpha \varepsilon J_n}$ of abelian projections in W' such that card. $J_n = n$, $Q_n = \sum_{\alpha \varepsilon J_n} E_\alpha'$ and $C_{E'} = Q_n$. As Q_n is countably decomposable in W', it follows that J_n is countable so that $n \leq \aleph_0$. Consequently, $Q_n = 0$ for $n > \aleph_0$.

(vi)$\Longrightarrow$(vii) Let P be a non-zero projection of UH-multiplicity n. By (vi) P is countably decomposable in W' and hence in W. By Proposition 1.3, there exists a unique Q_n such that $P \leq Q_n$. As in the proof of (v)$\Longrightarrow$(vi) we note that $Q_k = 0$ for $k > \aleph_0$ and hence $n \leq \aleph_0$. Consequently, by Theorem 2.6, (vii) holds.

(vii)$\Rightarrow$(viii) By Proposition 1.3 Q_n has UH-multiplicity n if $Q_n \neq 0$. Then by (vii), $n \leq \aleph_0$ if $Q_n \neq 0$ and hence $Q_n = 0$ for $n > \aleph_0$. Again by (vii) as Q_n has UOSD-multiplicity n for $n \in J_0 = \{n: Q_n \neq 0\}$ there exists an orthonormal set $\{x_{nj}\}_{j=1}^{n}$ in $Q_n H$ such that $Q_n H = [E(\sigma) x_{nj}: \sigma \in \mathbf{S}, j=1,2..,n]$. Therefore, $H = [E(\sigma) x_{nj}, n \in J_0, j=1,2,...,n]$ and hence (viii) holds.

(viii)$\Rightarrow$(i) By (viii) and by Lemma 3.3.9 of [3] W' is countably decomposable and hence W is countably decomposable. Besides, evidently for every projection P of UH- multiplicity n in W, $n \leq \aleph_0$. Thus (i) holds.

4. COMPARISON BETWEEN OSD-MULTIPLICITY AND H-MULTIPLICITY.

Example 2.4 is just a particular case of the following more general result.

THEOREM 4.1. Suppose E(.) has the CGS-property in H. Let P be a non-zero projection in W with the H-multiplicity n and with the OSD-multiplicity (relative to E(.)) N. Then:

(i) $n \leq N$.

(ii) $n = N$ if and only if P has UH-multiplicity n.

(iii) $n = N$ if and only if P has UOSD-multiplicity n (relative to $E(.)$).

PROOF. By Theorem 62.4 of [2] there exists a non-zero projection Q in W such that $Q \leq P$ and such that Q has UH-multiplicity n. Besides, by Theorem 3.1, Q is countably decomposable in W and $n \leq \aleph_0$. Therefore, by Theorem 2.6 Q has the UOSD-multiplicity n relative to $E(.)$. Consequently, by Theorem 5. of [4] the total multiplicity of Q is n and therefore, $n \leq N$.

(ii) Suppose $n = N$. We discuss the following two cases.

Case 1. n is finite.

By hypothesis, there exists a maximal orthogonal family $\{E_i'\}_1^n$ of abelian projections in W' such that $C_{E_i'} = P$ for all i. If P does not have UH-multiplicity then by Theorem 2.2 $P \neq \sum_1^n E_i'$. Now, by Theorem 3.1 (iii) and Lemma 2.5 there exist vectors $x_i \in PH$ such that $E_i' = [Wx_i]$, $i=1,2,\ldots,n$. Then by Theorems 66.2 and 65.2 of [2] we have $\rho(x_1) \equiv \rho(x_2) \equiv \equiv \ldots \equiv \rho(x_n)$. Let $E' = \sum_1^n E_i'$. Clearly, $E' \in W'$ and $E'H = \sum_1^n \oplus Z(x_i)$ is a UOSD of $E'H$ relative to $E(.)E'$. If $x \in (P-E')H$ and $x \neq 0$, then $[Wx] = Z(x) \perp E'H$ and $C_{[Wx]} \leq P = C_{[Wx_1]}$. Consequently, by Theorem 65.2 of [2] we conclude that $\rho(x) \ll \rho(x_1)$. On the other hand, by Theorem 1 of [4] there exists an OSD: $(P-E')H = \sum_{n+1}^{\ell} \oplus Z(x_i)$, $\ell \in \mathbb{N} \cup \{\infty\}$, of $(P-E')H$ relative to $E(.)(P-E')$ so that $PH = \sum_{1}^{\ell} \oplus Z(x_i)$ is an OSD of PH relative to $E(.)P$. Thus $\ell = N$ and P has OSD-multiplicity $N > n$. This contradiction proves that P has UH-multiplicity n.

Case 2. n is infinite.

Due to Theorem 3.1, $n = \aleph_0$ and $Q_k = 0$ for $k > \aleph_0$. Since $P = \sum_{\ell \leq \dim H} PQ_\ell = \sum_{\ell \leq \aleph_0} PQ_\ell$ and since the H-multiplicity of P is $\aleph_0$ and is given by $\min\{\ell : PQ_\ell \neq 0\}$, we have $pQ_\ell = 0$ for $\ell \neq \aleph_0$. Thus $P \leq Q_{\aleph_0}$ and hence P has UH-multiplicity $\aleph_0$ by Proposition 1.3.

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