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CONSTRAINED CONTROLLABILITY IN NON REFLEXIVE BANACH SPACES

BY

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Constrained Controllability in Non Reflexive Banach Spaces

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A Preliminary Report

In many problems concerning controllability and accessibility of Banach space valued systems, hypotheses on the Banach spaces such as separable and reflexive arise naturally; frequently, such hypotheses are employed in one capacity or another at each stage of studying a given control system. It is the purpose of this note to show how some tools from Banach space theory can be used to painlessly remove the reflexivity from conditions like "separable and reflexive" thereby rendering considerable generality to the resulting conclusions.

We start by outlining the main construction of G. Peichl and W. Schappacher [1986]/D. Barcenas and H. Leiva [1989]. Next, we gather some "tools of the trade" from Banach space theory, tailoring these results to the ends in mind. Lastly, we show how these tools can be systematically applied to erase the "reflexivity" from the "separable, reflexive" hypotheses used in our opening paragraphs.

We believe that non-specialists in Banach space theory can benefit from some of the procedures we discuss herein. The results invoked are very general, *yet sharp*, and, since we remove the more subtle of the two hypotheses, "separable and reflexive", these results are worthy of the non-specialists' attention.

§1. A. Review

It is well-known (E. Lee and L. Markus [1967]) that if $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, A and B are $n \times n$ and $m \times n$ matrices, respectively, and Ω is a compact subset of U, then the set of accessible points for the linear system

$$\begin{cases} \dot{x}(t) & = Ax(t) + Bu(t) \\ x(0) & = x_o \end{cases}$$

with measurable controls u taking values in Ω, is compact and convex.

G. Peichl and W. Schappacher [1986] have extended this result to the case where U is a reflexive Banach space, Ω is a closed bounded convex set in U and concluded that the set of accessible points is weakly compact and convex. D. Barcenas and H. Leiva [1989] went on to extend the Peichl-Schappacher result to the case U and U^* have the Radon-Nikodym property; their proof highlights several aspects of reflexivity not made plain by the Peichl-Schappacher work. It was, then, a combination of these two works which led to the present note. Since the purpose of this note is, in part, to demonstrate the use of Banach space techniques to relax reflexivity hypotheses, we will outline the Peichl-Schappacher/Barcenas-Leiva proof in a reflexive setting.

§1. The opening scene.

Let's get our bearings.

Let U and X be **reflexive** Banach spaces, U separable. consider the linear system

$$\begin{cases} \dot{x}(t) & = Ax(t) + Bu(t) \\ \\ x(0) & = x_o \in \mathcal{D}(A) \end{cases} \tag{$*$}$$

where $A : D(A) \subseteq X \to X$ is a linear operator generating a strongly continuous semigroup $(S_t)_{t\geq 0}$, $B : U \to X$ is a bounded linear operator and $u : [0, \infty) \to U$ is a strongly measurable essentially bounded function.

We suppose that Ω is a weakly compact, convex subset of U.

We call on some basic terminology from optimal control theory (see R. Curtain and A. Pritchard [1978] and G. Peichl and W. Schappacher [1986]).

The set

$$\tilde{\Omega}_T = \{u \in L_U^\infty[0, T] : u \in \Omega \ a.e.\}$$

is called the set of **admissible controls** of $(*)$, while the set

$$A_T(x_o) = \left\{ S_T x_o + \int_0^T S_{T-t} Bu(t) dt : u \in \tilde{\Omega}_T \right\}$$

is called the set of **accessible points** of $(*)$. The system $(*)$ is **controllable** if $0 \in A_T(x_o)$.

<u>**Proposition 1**</u>. *$\tilde{\Omega}_T$ is a weakly compact subset of $L_U^p[0,T]$* for $1 \leq p < \infty$ *and $A_T(x_o)$ is a weakly compact subset of X.*

The procedure of Peichl-Schappacher/Barcenas-Leiva is roughly the following; we've assumed U is separable and reflexive so, if $1 \leq p < \infty$, $\tilde{\Omega}_T$ *is weakly compact and convex* in $L_U^p[0,T]$ by the classical Dunford-Pettis theorem (suitably modified to the vector-valued case; see J. Diestel and J.J. Uhl, Jr. [1977]). But the operator

$$C : L_U^p[0,T] \to X$$

defined by

$$Cu = \int_0^T S_{T-s}Bu(s)ds$$

is a bounded linear operator and so takes the weakly compact, convex set $\tilde{\Omega}_T$ to a weakly compact convex subset of X. It is plain that

$A_T(x_o)$ is but a linear translate of said set and so *$A_T(x_o)$ is weakly compact and convex, too.*

We want to remark here that the reflexivity of U allows one to conclude that $L^p_U[0,T]$ is reflexive, too; though this seems straight forward, it is **not**. Indeed, one needs to know that U's reflexivity allows one to apply a vector-valued version of the Lebesgue-Vitali theorem (with functions taking values in U^*) to realize that $L^p_U[0,T]^* = L^{p^*}_{U^*}[0,T]$ where $\frac{1}{p} + \frac{1}{p^*} = 1$. Further, we take note of our inclusion (first noticed by Barcenas and Leiva) of $p = 1$ in this scheme; by the same reasoning just hinted, $L^1_U[0,T]^*$ is $L^\infty_{U^*}[0,T]$ and so with a bit of care one can show that $K \subset L^1_U[0,T]$ is relatively weakly compact if and only if K is uniformly integrable, following almost exactly the classical Dunford-Pettis pattern to proving this theorem. Of course, $\tilde{\Omega}_T$, being uniformly essentially bounded, is uniformly integrable.

Next Peichl-Schappacher/Barcenas-Leiva employ the weak compactness (and only that) of Ω to conclude to the following.

Proposition 2. (i) *For each $s \in [0,T]$, define the map $F_s : X^* \to [0,\infty)$ by*

$$F_s(x^*) := \max_{v\in\Omega} < x^*, S_s Bv > .$$

Then $\{F_s : s \in [0,T]\}$ is equicontinuous.

(ii) *For each $x^* \in X^*$, the mapping of $[0,T]$ to $[0,\infty)$ that takes $s \in [0,T]$ to $\max_{v\in\Omega} < x^*, S_s Bv >$ is continuous.*

Next, a measurable selection theorem (due to N.U. Ahmed and K.L. Teo [1981]) proves useful.

Proposition 3. *Denote by $WC(U)$ the collection of non-empty weakly compact subsets of U and let K be a compact subset of $\mathbb{R}$. Suppose $\Gamma : K \to WC(U)$ satisfies*

$$(1) \qquad \bigcup_{t\in K} \Gamma(t) \text{ is bounded in } U$$

and

(2) for any sequence (t_n) in K if $t^* = \lim_n t_n$

then

$$\bigcap_n \overline{\bigcup_{i \geq n} \Gamma(t_i)}^{\text{weak}} \subseteq \Gamma(t^*)$$

Then there exists a strongly measurable $u : K \to U$ *such that* $u(t) \in \Gamma(t)$ *a.e. in* K.

A consequence:

Proposition 4. *For each* $x^* \in X^*$ *there is* $u \in \tilde{\Omega}_T$ *such that* $< x^*, S_s Bu >= \max_{v \in \Omega} < x^*, S_s Bv >$ *for almost every* $s \in [0, T]$.

The separable reflexive nature of U comes into play in Proposition 3's proof in two essential places: first, to deduce that U^* is separable, thereby providing a norming sequence against which one can optimize and diagonalize; second, on finding a scalarly measurable selection with values in U, an appeal is made to Pettis's measurability theorem to deduce that said selection is, in fact, strongly measurable.

The upshot of these considerations is the following elegantly formulated result.

Theorem 1 (2.3 of Peichl-Schappacher). *Let X, U be reflexive Banach spaces with U separable. Let $B : U \rightarrow X$ be a bounded linear operator, A be the infinitesimal generator of a C_o-semigroup $(S_s)_{s\geq 0}$ of operators on X and Ω be a weakly compact convex subset of U that contains 0. Then for each $T > 0$, $0 \in A_T(x_o)$ if and only if for each $x^* \in X^*, < x^*, S_T x_o > + \int_0^T \max_{\nu \in \Omega} < x^*, S_t Bv > dt \geq 0$.*

§2. **The plot.**

We're interested in removing the reflexivity hypotheses from the results cited in §1. That such is possible is due to the very special character of weakly compact sets in general Banach spaces. Of course, weakly closed bounded subsets of reflexive spaces are weakly compact but the converse is also so, in a sense. The following result of W.J. Davis, T. Figiel, W.B. Johnson and A. Pelczynski [1974] makes precise the sense in which weakly compact sets **live** in reflexive spaces.

The Davis-Figiel-Johnson-Pelczynski Factorization Scheme.

Let K be a non-empty weakly compact subset of a Banach space Z. Then there exists a reflexive Banach space R and a bounded 1-1 linear operator $F : R \to Z$ such that $FB_R \supseteq K$.

While the result appears to be very imposing, in fact, **it's original proof is not difficult** though it is cleverly achieved.

In our context, it is noteworthy that the operator F takes a weakly compact set C inside the closed unit ball B_R of the reflexive space R in an **bijective weakly continuous** fashion onto the given weakly compact set K. F is therefore an affine homeomorphism between (C, weak) and (K, weak). This feature of the factorization scheme reaps manifold benefits one of which was noted by J. Diestel [1977] and is particularly relevant to the present discussion.

Proposition 5. *Let K be non-empty weakly compact, convex subset of the Banach space Z. Suppose $(\Omega, \sum, \mu)$ is a finite measure space and $1 \leq p < \infty$. Then the set $\{f \in L_Z^p(\mu) : f(w) \in K \ \mu - a.e.\}$ is a weakly compact convex subset of $L_Z^p(\mu)$.*

Though this result suffices for our present purposes, we would be remiss if we didn't mention that recently the first general criteria for relative weak compactness in $L_Z^p(\mu)$, general Z and $1 \leq p < \infty$, have been uncovered by A. Ülger [1991] and J. Diestel, W. Ruess and W. Schachermayer [1993].

The result:

Proposition 6. *A bounded subset K of $L_Z^p(\mu)$ is relatively weakly compact if and only if K is uniformly integrable and given any se-*

quence $(f_n) \subseteq K$ *there is a sequence* (g_n) *such that* $g_n \in$ *conv* $\{f_n, f_{n+1}, \ldots\}$ *and* $(g_n(\omega))$ *converges in* $(Z, \| \ \|)$ *for* μ*-almost all* $\omega \in \Omega$. *This happens if and only if* K *is uniformly integrable and given a sequence* $(f_n) \subseteq K$ *there is a sequence* (g_n) *such that* $g_n \in$ *conv* $\{f_n, f_{n+1}, \ldots\}$ *and* $(g_n(\omega))$ *converges weakly in* Z *for almost every* $\omega \in \Omega$.

Since bounded sets in $L_Z^p(\mu)$ are uniformly integrable in case $p > 1$, the condition of uniform integrability in these cases is redundant. We also hasten to add that A. Ülger [1991] showed how to deduce Proposition 5 from Proposition 6 in short order, without recourse to the Factorization Scheme of Davis, et al.

§3. Scene two.

We return to §1's results and reformulate them (in considerably greater generality). *We suppose* U *and* X *are Banach spaces. Con-*

sider the linear system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ \\ x(0) &= x_o \in D(A) \end{cases} \qquad (**)$$

where $A : X \to X$ is a linear operator generating a strongly continuous semigroup $(S_t)_{t\geq 0}$ whose dual semigroup $(S_t^*)_{t\geq 0}$ is also strongly continuous in $(0,\infty)$, $B : U \to X$ is a bounded linear operator and $u : [0,\infty) \to U$ is a strongly measurable essentially bounded function. We suppose that Ω is a non-empty separable weakly compact convex subset of U.

Proposition 5 says we can state Proposition 1 in the form

Proposition 1′. *$\tilde{\Omega}_T$ is a weakly compact convex subset of $L^p_U[0,T]$ for each $1 \leq p < \infty$ and $A_T(x_o)$ is a weakly compact convex subset of X.*

Proposition 2 called on Ω's weak compactness without regards for the structure of the ambient space U so it is still applicable in our present set-up. Proposition 3 can be amended as follows.

Proposition 3′. *Let U be any Banach space and Ω be a non-empty separable weakly compact convex subset of U. Let K be a non-empty compact subset of $\mathbb{R}$ and Λ be a map from K into the collection of non-empty weakly closed subsets of Ω such that*

$$\bigcap_{n}\overline{\bigcup_{n\geq i}\Lambda(t_i)}^{\text{weak}} \subseteq \Lambda(t^*)$$

whenever $(t_i) \subseteq K$ and $t_i \to t^*$. Then there is a strongly measurable function $g : K \to \Omega$ such that $g(t) \in \Lambda(t)$ for (almost) every $t \in K$.

Were U reflexive, Proposition 3 would be in play. Generally, we call on the Factorization Scheme. Start with the non-empty separable weakly compact convex subset Ω of the Banach space U. We can assume U is separable. Apply the Factorization Scheme to find a reflexive Banach space R, an injective bounded linear operator $F : R \to U$ and a weakly closed subset C of B_R such that $F(C) = \Omega$. F is a weak homeomorphism of C and Ω, Ω is weakly separable so C

is; it follows that the closed linear span of C is separable by Mazur's theorem, so we can assume R is a **separable** reflexive Banach space.

Define $\Gamma : K \to WC(R)$ by $\Gamma(t) = F^{-1}(\Lambda(t))$. Since F (and hence F^{-1}) is a weak homeomorphism, should (t_n) be a sequence in K with $t_n \to t^*$ we have

$$\begin{aligned}
\textstyle\bigcap_n \overline{\bigcup_{n\geq i} \Gamma(t_i)}^{\text{weak}} &= \textstyle\bigcap_n \overline{\bigcup_{n\geq i} F^{-1}(\Lambda(t_i))}^{\text{weak}} \\
&= \textstyle\bigcap_n \overline{F^{-1} \bigcup_{n\geq i}(\Lambda(t_i))}^{\text{weak}} \\
&= \textstyle\bigcap_n F^{-1}\overline{\bigcup_{n\geq i} \Lambda(t_i)}^{\text{weak}} \\
&= F^{-1}(\textstyle\bigcap_n \overline{\bigcup_{n\geq i} \Lambda(t_i)}^{\text{weak}}) \\
&= F^{-1}(\Lambda(t^*)) = \Gamma(t^*).
\end{aligned}$$

A strongly measurable selection $f : K \to C$ results with $f(t) \in \Gamma(t) = F^{-1}(\Lambda(t))$ holding for (almost) all $t \in K$.

Remember $F : R \to U$ is a bounded linear operator so $F \circ f = g : K \to \Omega$ is strongly measurable and is plainly a selection for Λ.

Proposition $3'$ in hand Proposition 4 now holds in our present non-reflexive context. All is set to piece Proposition $1'$, Proposition

2, Proposition 3′ and Proposition 4 together more-or-less as Peichl and Schappacher did Proposition 1,2,3 and 4. The result:

Theorem 1′. *Let X and U be Banach spaces, let $B : U \to X$ be a bounded linear operator and $A : X \to X$ be the infinitesimal generator of a C_o-semigroup $(S_t)_{t\geq 0}$ on X whose dual semigroup is strongly continuous on $(0,\infty)$. Suppose Ω is a non-empty* **separable**, *weakly compact convex subset of U containing 0. Then for each $T > 0$, $0 \in A_T(x_o)$ if and only if for each $x^* \in X^*$,*

$$< x^*, S_T x_o > + \int_o^T \max_{v\in\Omega} < x^*, S_t B v > dt \geq 0.$$

An interesting point to the comments above is that we've reduced the question of accessibility of controls to a problem in semi groups of operators, namely, given a C_o-semigroup $(S_t)_{t\geq 0}$ of operators on a Banach space X under what conditions is the dual semigroup strongly continuous on $(0,\infty)$? This question is taken up in the sequel.

To conclude, we'd like to address the issue of the strong continuity of $(S_t^*)_{t>0}$. To give our remarks some semblence of coherency we call on an old friend: the Gelfand integral.

Suppose $(\Omega, \sum, P)$ is a complete probability space (any finite measure space will do as well, actually), and X is a Banach space. A function $f : \Omega \to X^*$ is said to be Gelfand integrable if for each $x \in X$, $f(\cdot)(x) \in L^1(P)$. It follows from the closed graph theorem that if $f : \Omega \to X^*$ is Gelfand integrable then for each event E in $\sum$ there is a unique $x_E^* \in X^*$ so that for each $x \in X$

$$x_E^*(x) = \int_E F(w)(x)dP(w).$$

The vector x_E^* is called the Gelfand integral of f over E and denoted by

$$G - \int_E f dp.$$

The point to be made here is that the Gelfand integral exists *under absolutely minimal conditions.*

An important thing to realize is that the Gelfand integral helps to describe any bounded linear operator from $L^1(P)$ to X^*. In fact, if $f : \Omega \to X^*$ is Gelfand integrable and essentially bounded, then for each $g \in L^1(P)$, $f(\cdot)g(\cdot)$ is Gelfand integrable and the operation

$$g \mapsto G\int_{\Omega} gf\, dP.$$

defines a bounded linear operator from $L^1(P)$ to X^*. The converse also obtains thanks to the lifting theorem: if $u : L^1(P) \to X^*$ is a bounded linear operator and $\lambda : L^\infty(P) \to L^\infty(P)$ is a lifting, then $f(\cdot)x = \lambda \circ u^*(\ )(x)$ defines an essentially bounded Gelfand integrable function such that for any $g \in L^1(P)$,

$$u(g) = G\int_{\Omega} gf dP.$$

Another ingredient is called for. Recall that a Banach space X has the Radon-Nikodym property if given a complete probability space $(\Omega, \sum, P)$ (again, any finite measure space will do) and any operator $u : L^1(P) \to X$ there is a Bochner integrable $f : \Omega \to X$

such that (f is essentially bounded and) for each $g \in L^1(P)$,

$$ug = \text{ Bochner } \int gfdP.$$

It is a remarkable theorem of N. Dunford and B.J. Pettis [1940] that says that if X^* is separable then X^* has the Radon-Nikodym property. It is an insightful observation of J.J. Uhl [1972] that if every separable subspace of X has a separable dual, then X^* has the Radon-Nikodym property. It is a stunning success of C. Stegall [1975] that says if X^* has the Radon-Nikodym property, then every separable subspace of X has a separable dual.

For our purposes we say that a Banach space X is an Asplund space if and only if X^* has the Radon-Nikodym property. This is purely a convenience and we warn the reader that we are condensing an enormous amount of deep and beautiful theory relating measure theory and the geometry of Banach spaces; we invite said-reader to look to the papers of C. Stegall, I. Namioka and R.R. Phelps cited in the bibliography to get a feel for what we've been discussing.

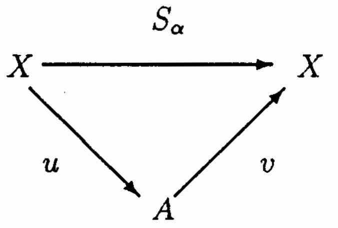
A bounded linear operator $u : X \to Y$ is called an **Asplund operator** if there exist bounded linear operators $v : X \to A$ and $w : A \to Y$ so that $u = wv$, where A is an Asplund space. This notion was introduced and successfully studied by C. Stegall [1981] who showed that an operator $u : X \to Y$ is an Asplund operator if and only if its adjoint $u^* : Y^* \to X^*$ factors through a dual space with the Radon-Nikodym property, that is, there is an Asplund space B so that for some bounded linear operators $W : Y^* \to B^*$ and $V : B^* \to X^*$ we have $u^* = VW$.

Here's our small contribution to the question of when the adjoint semigroup of $(S_t)_{t \geq 0}$ is strongly continuous on $t > 0$. It is inspired by the result of J.M.A.M. van Neerven [1990] and D. Barcenas - H. Leiva (unpublished) but is more widely applicable because it addresses each semi-group candidate individually. We follow van

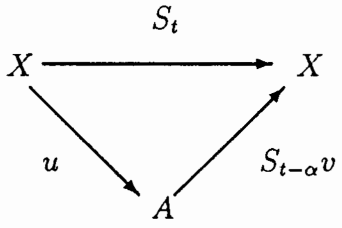
Neerven's [1990] proof.

Theorem. *Suppose $(S_t)_{t\geq 0}$ is a strongly continuous semigroup of operators on the Banach space X and that for each $t > 0$, S_t is an Asplund operator. Then $(S_t^*)_{t>0}$ is strongly continuous.*

Proof: Let $[\alpha, \beta]$ be any closed bounded unit interval contained in $(0, \infty)$. Since S_α is an Asplund operator there is an Asplund space A and bounded linear operator $u : X \to A$ and $v : A \to X$ so that the following diagram is commutative



Notice that for any $t \in [\alpha, \beta]$, the following diagram also commutes, thanks to the semigroup property



Everything is in place. Take $x^* \in X^*$ and for $t \in [\alpha, \beta]$ define

$$f_{x^*}(t) = v^* S^*_{t-\alpha}(x^*).$$

It is easy to see that $f_{x^*} : [\alpha, \beta] \to A^*$ is weak* continuous hence Gelfand integrable with respect to Lebesgue measure on the unit interval $[\alpha, \beta]$. An operator $T : L^1[\alpha, \beta] \to A^*$ is born:

$$Tg = G - \int_\alpha^\beta g(t) f_{x^*}(t) dt$$

But A is an Asplund space and so A^* enjoys the Radon-Nikodym property thanks to MESSRS Namioka and Phelps. It follows that T is representable in the form

$$Tg = \text{ Bochner } \int_\alpha^\beta g(t) h(t) dt$$

for some essentially bounded, strongly measurable $h : [\alpha, \beta] \to X^*$.

Here's the situation: for $g \in L^1[\alpha, \beta]$

$$G \int_\alpha^\beta g(t) f_{x^*}(t) dt = Tg = B \int_\alpha^\beta g(t) h(t) dt.$$

Look to $g(s) = \chi_{[t,t+\epsilon)}(s)$; on normalization we have

$$G\int_t^{t+\epsilon} \frac{1}{\epsilon} f_{x^*}(s)ds = T\left(\chi_{\frac{[t,t+\epsilon)}{\epsilon}}\right) = B\int_t^{t+\epsilon} \frac{1}{\epsilon} h(s)ds.$$

Let $\epsilon \to 0$ and look closely. Because f_{x^*} is weak* continuous, the left hand side tends (weak*) to $f_{x^*}(t)$. By Lebesgue's theorem (which applies to the Bochner integral), the right side tends (in norm) to $h(t)$ for almost all $t \in [\alpha, \beta]$. So $f_{x^*}(t) = h(t)$ for almost all $t \in [\alpha, \beta]$. It follows that for each $x^* \in x^*$

$$t \longrightarrow (S_{t-\alpha}v)^*(x^*)$$

is strongly measurable on $[\alpha, \beta]$. From this we see that $t \to u^*(S_{t-\alpha}v)^*(x^*) = (S_{t-\alpha}vu)^*(x^*) = (S_{t-\alpha}S_\alpha)^*(x^*) = S_t^*(x^*)$ is too. We conclude that $t \to S_t^*x^*$ is strongly measurable on $(0, \infty)$ and so the still useful old gem of E. Hille assures us of the strong continuity of $(S_t^*)_{t>0}$.

The easiest way to assure each S_t is an Asplund operator is to suppose X^* has the Radon-Nikodym property (or equivalently, each separable subspace of X has a separable dual); this is the result of van Neerven [1990] and Barcenas-Leiva.

In case X^* does not have the Radon-Nikodym property it may still happen that each S_t is Asplund. Here's how to recognize that such is the case in some special spaces whose duals don't have the Radon-Nikodym property.

Example 1. If $X = C(K)$, then X^* has the Radon-Nikodym property if and only if K contains no perfect subsets. Many of the $C(K)$'s that arise in the study of control systems are based on K's (like $[0,1]$, $[0,1]^2$, etc.) that **do** have perfect subsets. In any case, *an operator $u : C(K) \to C(K)$ is an Asplund operator if and only if given a bounded sequence (f_n) in $C(K)$, its image (uf_n) has a* **pointwise** *convergent subsequence.* This we feel is a condition which is easily tested and so might be of use to non Banach space specialists.

Example 2. *If X is a weakly sequentially complete Banach space,* *then an operator $u : X \to X$ is an Asplund operator precisely when u is a weakly compact operator, i.e., $u\,B_X$ is relatively weakly compact in X.*

All the Lebesgue spaces $L_p(\mu)(1 \leq p < \infty)$ are weakly sequentially complete, as are the (weighted) Lorentz spaces $L_{W,P}(1 \leq p < \infty)$ of G.G. Lorentz [1950]; the classical Lorentz spaces $L_{p,q}(1 \leq q < p < \infty)$ and Orlicz spaces $L_\Phi(\mu)$ generated by an Orlicz function Φ that satisfies the Δ_2-condition.

In each of these cases, the space under consideration is a Banach function space and so weakly compact sets in these spaces are understood. In case of the Lebesgue spaces, N. Dunford and J.T. Schwartz [1958] tells all about weak compactness; for Lorentz spaces, the [1980] dissertation of J. Creekmore should be consulted while for Orlicz spaces, J. Alexopoulos [1999] is worth a look.

In case the measure on which the function space is modeled is finite (so for $L_1[0,1]$, $L_{p,q}[0,1]$ of $L_\Phi[0,1]$), then a bounded subset K of the function space is relatively weakly compact precisely when $g \cdot K$ is uniformly integrable for each g in the Köthe dual of the space. Here the Köthe dual of F is the collection of all measurable g such that $\int |f \cdot g| < \infty$ for each $f \in F$.

Example 3. *If $(S_t)_{t>0}$ is a strongly continuous semi-group of Asplund operators on an L_1-space then each S_t must, in fact, be compact (and* **conversely***).*

This follows from the facts cited in Example 2, the fact that L_1-spaces have the Dunford-Pettis property of A. Grothendieck [1953], and an application of the semi-group property: $S_t = (S_{t/2})^2$.

We might remark that there are other important weakly sequentially complete Banach spaces that enjoy the Dunford-Pettis property; among them:

$$L_1(\mathbf{T})/H_0^1,\ C(\mathbf{T})/A,\ A^*,\ H^{\infty *}.$$

Example 4. *In case X is a Banach space with the Dunford-Pettis property and the Grothendieck property, then H.P. Lotz [1985] has discovered the remarkable fact that any strongly continuous semigroup $(S_t)_{t \geq 0}$ has a bounded infinitesimal generator and so $(S_t^*)_{t>0}$ is strongly continuous.*

Among the X's that are covered by the Lotz discovery are all $L_\infty(\mu)$-spaces and H^∞.

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