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V. Applications to integration in locally compact Hausdorff spaces-Part II

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The Bartle-Dunford-Schwartz integral V. Integration in locally compact Hausdorff spaces-Part II

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This part consists of Sections 20-24. The classical Lusin's theorem is generalized in Section 20 for $\sigma(\mathcal{P})$- measurable functions with respect to an X-valued σ-additive measure $\mathbf{m}$ defined on $\mathcal{P}$ where X is a Banach space or an lcHs and $\mathcal{P} = \mathcal{B}(T)$ (resp. $\mathcal{B}_c(T)$, $\mathcal{B}_0(T)$, $\delta(\mathcal{C})$, $\delta(\mathcal{C}_0)$) and it is deduced that $C_c(T)$ (resp. $C_0(T)$) is dense in $\mathcal{L}_p(\mathbf{m})$ and $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$, $1 \le p < \infty$ for both the cases of X when $\mathcal{P} = \delta(\mathcal{C})$ or $\delta(\mathcal{C}_0)$ (resp. when $\mathcal{P} = \mathcal{B}_0(T)$ or $\mathcal{B}_c(T)$ or $\mathcal{B}(T)$). Section 21 is devoted to the study of the Lusin measurability of functions and sets. Let $\mathbf{m} : \mathcal{B}(T) \to X$ (resp. $\mathbf{n} : \delta(\mathcal{C}) \to X$) be σ-additive and Borel regular (resp. and $\delta(\mathcal{C})$-regular). Then it is shown that that a scalar function f on T is Lusin $\mathbf{m}$-measurable if and only if it is $\mathbf{m}$-measurable (see Theorem 21.5); and it is $\mathbf{n}$-measurable if and only if it is Lusin $\mathbf{n}$-measurable with $N(f)$ being σ-bounded(see Theorem 21.6). Section 22 is devoted to improve Theorem 4.2 of [P8] and Theorem 12.2 of [P10] for $\mathbf{m}$ and $\mathbf{n}$. Section 23 is devoted to present the Baire version of Corollary T_2, Appendix I of [T] and to generalize it to σ-additive regular vector measures. Finally, Section 24 describes the duals of $\mathcal{L}_1(\mathbf{m})$ and $\mathcal{L}_1(\mathbf{n})$ when X is a Banach space and gives the vector measure analogues of Theorem 4.1 and Proposition 5.9 of [T]. Of course, some of the ideas and techniques found in [T] are suitably adapted in this study.

20. GENERALIZED LUSIN'S THEOREM AND ITS VARIANTS

In the sequel, T denotes a locally compact Hausdorff space and $\mathcal{U}$, $\mathcal{C}$, $\mathcal{C}_0$ are as in Definition 16.4 of [P10]. Then $\mathcal{B}(T) = \sigma(\mathcal{U})$, the σ-algebra of the Borel sets in T; $\mathcal{B}_c(T) = \sigma(\mathcal{C})$, the σ-ring of the σ-Borel sets in T and $\mathcal{B}_0(T) = \sigma(\mathcal{C}_0)$, the σ-ring of the Baire sets in T. $\delta(\mathcal{C})$ and $\delta(\mathcal{C}_0)$ denote the δ-rings generated by $\mathcal{C}$ and $\mathcal{C}_0$.

Notation 20.1. $C_c(T) = \{f : T \to \boldsymbol{K}, f \text{ continuous with compact support}\}$; $C_c^r(T) = \{f \in C_c(T) : f \text{ real}\}$; $C_c^+(T) = \{f \in C_c^r(T) : f \ge 0\}$; $C_0(T) = \{f : T \to \boldsymbol{K}, f \text{ is continuous and vanishes at infinity in } T\}$; $C_0^r(T) = \{f \in C_0(T) : f \text{ real}\}$ and $C_0^+(T) = \{f \in C_0^r(T) : f \ge 0\}$. All these spaces are provided with the supremum norm $||\cdot||_T$.

As in Parts I, III and IV, X denotes a Banach space or an lcHs over $\boldsymbol{K}$($\boldsymbol{R}$or $\boldsymbol{C}$) with Γ, the family of all continuous seminorms on X, unless otherwise mentioned and it will be explicitly specified whether X is a Banach space or an lcHs. Let $\mathcal{P} = \mathcal{B}(T)$(resp. $\mathcal{B}_c(T)$, $\mathcal{B}_0(T)$, $\delta(\mathcal{C})$, $\delta(\mathcal{C}_0)$) and let $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $\mathcal{P}$-regular (see Definition 16.7 of [P10]). In this section we obtain the generalized Lusin's theorem and its variants for $\sigma(\mathcal{P})$-measurable scalar functions on T, with respect to $\mathbf{m}$ when X is a Banach space and when X is an lcHs. Then we deduce that

$C_c(T)$ (resp. $C_0(T)$) is dense in $\mathcal{L}_p(\mathbf{m})$ and $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$, $1 \leq p < \infty$, for both the cases of X when $\mathcal{P} = \delta(\mathcal{C})$ or $\delta(\mathcal{C}_0)$ (resp. when $\mathcal{P} = \mathcal{B}_0(T)$ or $\mathcal{B}_c(T)$ or $\mathcal{B}(T)$).

Theorem 20.2(**Generalized Lusin's theorem for m on** $\mathcal{B}(T)$).

(i) (**m normed space-valued**). Let X be a normed space and $\mathbf{m} : \mathcal{B}(T) \to X$ be σ-additive and Borel regular. Suppose $f : T \to \boldsymbol{K}$ is Borel measurable. Then, given $\epsilon > 0$, there exists $g \in C_c(T)$ such that

$$||\mathbf{m}||(N(f-g)) = ||\mathbf{m}||(\{t \in T : f(t) - g(t) \neq 0\}) < \epsilon \qquad (20.2.1)$$

and

$$||g||_T \leq ||f||_T. \qquad (20.2.2)$$

(ii) (**m lcHs-valued**). Let X be an lcHs and let $\mathbf{m} : \mathcal{B}(T) \to X$ be σ-additive and Borel regular and let f be as in (i). Then, given $\epsilon > 0$ and $q \in \Gamma$, there exists $g_q \in C_c(T)$ such that

$$||\mathbf{m}||_q(N(f-g_q)) < \epsilon \qquad (20.2.3)$$

and

$$||g_q||_T \leq ||f||_T. \qquad (20.2.4)$$

Proof. (i) Let $\tilde{X}$ be the Banach completion of X. Then $\mathbf{m} : \mathcal{B}(T) \to X \subset \tilde{X}$ and hence $\mathbf{m}$ can be considered as Banach space valued. As $\mathbf{m}$ is Borel inner regular in T, there exists $K \in \mathcal{C}$ such that $||\mathbf{m}||(T\backslash K) < \frac{\epsilon}{2}$. By hypothesis, $f\chi_K$ is $\mathcal{B}(T)$-measurable and vanishes in $T\backslash K$. If $f\chi_K$ is bounded in T, then the proof of Theorem 2.23 of [Ru1] for the case of bounded Borel functions holds here if we replace μ by $||\mathbf{m}||$, since $||\mathbf{m}||$ is σ-subadditive on $\mathcal{B}(T)$ by Proposition 2.2 of [P8]. Hence there exists $g \in C_c(T)$ such that $||\mathbf{m}||(N(f\chi_K - g)) < \frac{\epsilon}{2}$. Then $||\mathbf{m}||(N(f-g)) \leq ||\mathbf{m}||(N(f\chi_K - g)) + ||\mathbf{m}||(T\backslash K) < \epsilon$. When $f\chi_K$ is unbounded, the argument in the proof of the said theorem of [Ru1] for the unbounded case also holds here since $||\mathbf{m}||$ is continuous on $\mathcal{B}(T)$ by Proposition 2.2 of [P8] and hence there exists $g \in C_c(T)$ such that $||\mathbf{m}||(N(f\chi_K - g)) < \frac{\epsilon}{2}$ so that by the above argument $||\mathbf{m}||(N(f-g)) < \epsilon$. Hence (20.2.1) holds.

To prove (20.2.2), it suffices to restrict to the case $||f||_T = M < \infty$. We argue as in the last part of the proof of the said theorem of [Ru1]. Let $g \in C_c(T)$ satisfy (20.2.1). Replacing g by $g_1 = \varphi \circ g$, where $\varphi(z) = z$ if $|z| \leq M$ and $\varphi(z) = \frac{Mz}{|z|}$ if $|z| > M$, we deduce that $g_1 \in C_c(T)$, $||\mathbf{m}||(N(f-g_1)) < \epsilon$ and $||g_1||_T \leq ||f||_T$. Hence (20.2.1) and (20.2.2) hold for g_1.

(ii) Given $q \in \Gamma$, $\mathbf{m}_q = \Pi_q \circ \mathbf{m} : \mathcal{B}(T) \to X_q \subset \widetilde{X_q}$ is σ-additive and Borel regular and hence by (i), (ii) holds.

To obtain the variants of Theorem 20.2 when $\mathbf{m} : \mathcal{R} \to X$ is σ-additive and $\mathcal{R}$-regular, where $\mathcal{R} = \mathcal{B}_c(T)$ (resp. $\mathcal{B}_0(T)$, $\delta(\mathcal{C})$, $\delta(\mathcal{C}_0)$), we give the following lemmas.

Lemma 20.3. Let X be a normed space or an lcHs. Then an X-valued σ-additive measure on $\mathcal{B}_0(T)$ (resp. on $\delta(\mathcal{C}_0)$) is $\mathcal{B}_0(T)$-regular (resp. $\delta(\mathcal{C}_0)$-regular).

Proof. The result for $\delta(\mathcal{C}_0)$ holds by Theorem in [DL] while that for $\mathcal{B}_0(T)$ holds by Remark on pp.93-94 of [DL].

Lemma 20.4.

(i) Let X be a normed space and let $\mathbf{n}_c : \delta(\mathcal{C}) \to X$ (resp. $\mathbf{n}_0 : \delta(\mathcal{C}_0) \to X$) be σ-additive and let $\mathbf{n}_c$ be $\delta(\mathcal{C})$-regular. If $f : T \to \boldsymbol{K}$ is $\mathcal{B}_c(T)$-measurable (resp. $\mathcal{B}_0(T)$-measurable) and if A is a compact in T such that $f(t) = 0$ for $t \in T\backslash A$, then, given $\epsilon > 0$, there exists $g \in C_c(T)$ such that

$$||\mathbf{n}_c||(N(f-g)) < \epsilon \quad (20.4.1) \quad (\text{resp.}||\mathbf{n}_0||(N(f-g)) < \epsilon \quad (20.4.2))$$

and moreover, we can choose $g \in C_c(T)$ such that

$$||g||_T \le ||f||_T. \quad (20.4.3)$$

(ii) If X is an lcHs in (i) and if the remaining hypotehsis of $\mathbf{n}_c$ (resp. $\mathbf{n}_0$) and of f remain the same, then, given $q \in \Gamma$ and $\epsilon > 0$, there exists $g_q \in C_c(T)$ such that

$$||\mathbf{n}_c||_q(N(f-g_q)) < \epsilon \quad (20.4.4) \quad (\text{resp.}||\mathbf{n}_0||_q(N(f-g_q)) < \epsilon \quad (20.4.5))$$

and moreover, we can choose $g_q \in C_c(T)$ such that

$$||g_q||_T \le ||f||_T. \quad (20.4.6)$$

Proof. (i) One can adapt the proof of Theorem 2.23 of [Ru1] as follows. Choose a relatively compact open set V such that $A \subset V$. In the construction of the functions on p.53 of [Ru1], we can observe that $2^n t_n$ (in the notation of [Ru1]) is the characteristic function of some σ-Borel (resp. Baire) set $T_n \subset A$ and

$$f(x) = \sum_1^\infty t_n(x),\ x \in T$$

since f is $\mathcal{B}_c(T)$-measurable (resp. $\mathcal{B}_0(T)$-measurable). By hypothesis, $\mathbf{n}_c$ is $\delta(\mathcal{C})$-regular (resp. by Lemma 20.3, $\mathbf{n}_0$ is $\delta(\mathcal{C}_0)$-regular) and hence there exist $K_n \in \mathcal{C}$ (resp. $K_n \in \mathcal{C}_0$) and an open set $V_n \in \delta(\mathcal{C})$ (resp. $V_n \in \delta(\mathcal{C}_0)$) such that $K_n \subset T_n \subset V_n \subset V$ with $||\mathbf{n}_c||(V_n\backslash K_n) < \frac{\epsilon}{2^n}$ (resp. with $||\mathbf{n}_0||(V_n\backslash K_n) < \frac{\epsilon}{2^n}$) for $n \in \mathbb{N}$. Let us suppose that $0 \le f \le 1$ in A. Then choosing h_n by Urysohn's lemma such that $K_n \prec h_n \prec V_n$ for all n and then defining $g(x) = \sum_1^\infty 2^{-n} h_n(x),\ x \in$

T as on p.54 of [Ru1] and using the fact that $||\mathbf{n}_c||$ (resp. $||\mathbf{n}_0||$) is σ-subadditive on $\mathcal{B}_c(T)$ (resp. on $\mathcal{B}_0(T)$), we note that $g \in C_c(T)$ and $||\mathbf{n}_c||(N(f-g)) < \epsilon$ (resp. and $||\mathbf{n}_0||(N(f-g)) < \epsilon$) and hence (20.4.1) (resp. (20.4.2)) holds. From this it follows that these inequalities hold if f is bounded. When f is not bounded, let $B_n = \{x : |f(x)| > n\}$. Then $B_n \searrow \emptyset$ in $\mathcal{B}_c(T)$ (resp. in $\mathcal{B}_0(T)$) and by hypothesis, B_n is relatively compact for all n. Then by Lemma 18.2 of [P11], $(B_n)_1^\infty \subset \delta(\mathcal{C})$ (resp. $(B_n)_1^\infty \subset \delta(\mathcal{C}_0)$). Since $X \subset \widetilde{X}$, the Banach completion of X, we can consider $\mathbf{n}_c$ and $\mathbf{n}_0$ as Banach space-valued and hence Proposition 2.2 of [P8], $||\mathbf{n}_c||(B_n) \to 0$ (resp. $||\mathbf{n}_0||(B_n) \to 0$). Then arguing as in the general case of Theorem 2.23 of [Ru1] with $||\mathbf{n}_c||$ (resp. $||\mathbf{n}_0||$) replacing μ we observe that (20.4.1) (resp. (20.4.2)) holds for the general case. (20.4.3) is proved as in the last part of the proof of Theorem 2.23 of [Ru1].

(ii) This is immediate from (i), since $(\mathbf{n}_c)_q : \delta(\mathcal{C}) \to X_q \subset \widetilde{X_q}$ is σ-additive and $\delta(\mathcal{C})$-regular and $(\mathbf{n}_0)_q : \delta(\mathcal{C}_0) \to X_q \subset \widetilde{X_q}$ is σ-additive for $q \in \Gamma$.

Lemma 20.5. Let X be an lcHs and let $\mathbf{m}_c : \mathcal{B}_c(T) \to X$ be σ-additive and σ-Borel regular. Then $\boldsymbol{\omega}_c = \mathbf{m}_c|_{\delta(\mathcal{C})}$ is σ-additive and $\delta(\mathcal{C})$-regular.

Proof. Clearly it suffices to prove the lemma when X is a normed space and hence let X be so. Since $\boldsymbol{\omega}_c$ is σ-additive, it suffces to prove the regularity of $\boldsymbol{\omega}_c$. Let $A \in \delta(\mathcal{C})$ and $\epsilon > 0$. Then by hypothesis, there exist $K \in \mathcal{C}$ and an open set $U \in \mathcal{B}_c(T)$ such that $K \subset A \subset U$ and $||\mathbf{n}_c||(U \backslash K) < \epsilon$. Since A is relatively compact, by Theorem 50.D of [H] there exists a relatively compact open set V such that $\bar{A} \subset V$. Then $W = U \cap V$ is an open set belonging to $\delta(\mathcal{C})$ by Lemma 18.2 of [P11], $K \subset A \subset W$ and $||\boldsymbol{\omega}_c||(W \backslash K) < \epsilon$. Hence the lemma holds.

Theorem 20.6 (Variants of the generalized Lusin's theorem). Let X be an lcHs. Let $\mathbf{m}_c : \mathcal{B}_c(T) \to X$ (resp. $\mathbf{n}_c : \delta(\mathcal{C}) \to X$, $\mathbf{m}_0 : \mathcal{B}_0(T) \to X$, $\mathbf{n}_0 : \delta(\mathcal{C}_0) \to X$) be σ-additive and let $\mathbf{m}_c$ be $\mathcal{B}_c(T)$-regular (resp. $\mathbf{n}_c$ be $\delta(\mathcal{C})$-regular). Suppose $f : T \to \boldsymbol{K}$ is $\mathcal{B}_c(T)$-measurable (resp. Baire measurable). Let $A \in \mathcal{B}_c(T)$ (resp. $A \in \delta(\mathcal{C})$, $A \in \mathcal{B}_0(T)$, $A \in \delta(\mathcal{C}_0)$) such that $f(t) = 0$ for $t \in T \backslash A$ and let $\epsilon > 0$. Then, given $q \in \Gamma$, there exists $g_q \in C_c(T)$ such that

$$||\boldsymbol{\omega}||_q(N(f - g_q)) < \epsilon \qquad (20.6.1)$$

where $\boldsymbol{\omega} = \mathbf{m}_c$ or $\mathbf{n}_c$ or $\mathbf{m}_0$ or $\mathbf{n}_0$, as the case may be. Moreover, $g_q \in C_c(T)$ can be chosen such that

$$||g_q||_T \leq ||f||_T. \qquad (20.6.2)$$

We say that $\mathbf{m}_c$ (resp. $\mathbf{m}_0$) is σ-Borel (resp. Baire) inner regular in T if, given $q \in \Gamma$ and $\epsilon > 0$, there exists $K \in \mathcal{C}$ (resp. $K \in \mathcal{C}_0$) such that $||\mathbf{m}_c||_q(B) < \epsilon$ for $B \in \mathcal{B}_c(T)$ (resp. $B \in \mathcal{B}_0(T)$) with $B \subset T \backslash K$. If $\mathbf{m}_c$ (resp. $\mathbf{m}_0$) is further σ-Borel (resp. Baire) inner regular in T, then the above results hold for any $\mathcal{B}_c(T)$-measurable (resp. $\mathcal{B}_0(T)$-measurable) function f on T with values in $\boldsymbol{K}$

Proof. Without loss of generality we shall assume X to be a normed space. Let $\mathcal{R} = \mathcal{B}_c(T)$ and $\boldsymbol{\omega} = \mathbf{m}_c$, or $\mathcal{R} = \delta(\mathcal{C})$ and $\boldsymbol{\omega} = \mathbf{n}_c$ or $\mathcal{R} = \mathcal{B}_0(T)$ and $\boldsymbol{\omega} = \mathbf{m}_0$ or $\mathcal{R} = \delta(\mathcal{C}_0)$ and $\boldsymbol{\omega} = \mathbf{n}_0$. By hypothesis and by Lemmas 20.3 and 20.5, $\boldsymbol{\omega}$ is $\mathcal{R}$-regular and σ-additive. Then there exists a compact set $K \in \mathcal{R}$ such that $K \subset A$ and $||\boldsymbol{\omega}||(A\backslash K) < \frac{\epsilon}{2}$. As $f\chi_K$ satisfies the hypothesis of Lemma 20.4(i) and as $\mathbf{m}_c|_{\delta(\mathcal{C})}$ and $\mathbf{m}_0|_{\delta(\mathcal{C}_0)}$ are regular and σ-additive by hypothesis and by Lemmas 20.5 and 20.3, by Lemma 20.4(i) there exists $g \in C_c(T)$ such that $||\boldsymbol{\omega}||(N(f\chi_K - g)) < \frac{\epsilon}{2}$ with $||g||_T \leq ||f\chi_K||_T \leq ||f||_T$. By hypothesis and by Theorem 51.B of [H], $f - g$ is $\sigma(\mathcal{R})$-measurable and hence $N(f-g) \in \sigma(\mathcal{R})$. Since $f(t)\chi_K(t) = f(t)$ for $t \in K \cup (T\backslash A)$, $N(f-g) \subset N(f\chi_K - g) \cap (K \cup (T\backslash A)) \cup (A\backslash K)$, and hence $||\boldsymbol{\omega}||(N(f-g)) < \epsilon$. Thus (20.6.1) and (20.6.2) hold.

If $\mathbf{m}_c$ is σ-Borel (resp. $\mathbf{m}_0$ is Baire) inner regular in T, choose $K \in \mathcal{C}$ (resp. $K \in \mathcal{C}_0$) such that $||\mathbf{m}_c||(B) < \frac{\epsilon}{2}$ (resp. $||\mathbf{m}_0||(B) < \frac{\epsilon}{2}$) for $B \in \mathcal{B}_c(T)$ (resp. $B \in \mathcal{B}_0(T)$) with $B \subset T\backslash K$. Let $\boldsymbol{\omega} = \mathbf{m}_c$ or $\mathbf{m}_0$ as the case may be. Then by the above part there exists $g \in C_c(T)$ such that $||\boldsymbol{\omega}||(N(f\chi_K - g)) < \frac{\epsilon}{2}$ with $||g||_T \leq ||f\chi_K||_T \leq ||f||_T$ and hence as in the proof of (20.2.1) of Theorem 20.2(i) we have $||\boldsymbol{\omega}||(N(f-g)) < \epsilon$.

Corollary 20.7. Let X be an lcHs and $q \in \Gamma$. Suppose $\mathbf{m} : \mathcal{B}(T) \to X$ is σ-additive and Borel regular (resp. $\mathbf{m}_c : \mathcal{B}_c(T) \to X$ is σ-additive and σ-Borel regular and moreover, σ-Borel inner regular in T, $\mathbf{m}_0 : \mathcal{B}_0(T) \to X$ is σ-additive and Baire inner regular in T). Let $f : T \to \boldsymbol{K}$ be Borel measurable (resp. σ-Borel measurable, Baire measurable). Then given $q \in \Gamma$, there exists a sequence $(g_n^{(q)}) \subset C_c(T)$ such that $\sup_n ||g_n^{(q)}||_T \leq ||f||_T$ and $f(t) = \lim_n g_n^{(q)}(t)$ $\mathbf{m}_q$-a.e. in T.

Proof. Without loss of generality we shall assume X to be a normed space. Let $\mathcal{R} = \mathcal{B}(T)$ (resp. $\mathcal{B}_c(T)$, $\mathcal{B}_0(T)$) and $\boldsymbol{\omega} = \mathbf{m}$ (resp. $\mathbf{m}_c$, $\mathbf{m}_0$). Then by Theorems 20.2 and 20.6 there exists $g_n \in C_c(T)$ with $||g_n||_T \leq ||f||_T$ such that $||\boldsymbol{\omega}||(N(f - g_n)) < \frac{1}{2^n}$ for $n \in \mathbb{N}$. Let $A_n = N(f - g_n)$ and let $A = \limsup_n A_n$. Clearly, $A \in \mathcal{R}$ and $||\boldsymbol{\omega}||(A) \leq ||\boldsymbol{\omega}||(\bigcup_{k \geq n} A_k) < \frac{1}{2^{n-1}} \to 0$ since $||\boldsymbol{\omega}||$ is σ-subadditive on $\mathcal{R}$. Hence $||\boldsymbol{\omega}||(A) = 0$. Clearly, $f(t) = \lim_n g_n(t)$ for $t \in T\backslash A$.

Lemma 20.8. Let X be a sequentially complete lcHs, $\mathcal{P} = \delta(\mathcal{C})$ or $\delta(\mathcal{C}_0)$ and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Then $C_c(T) \subset \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ for $1 \leq p < \infty$ (see Definition 14.4 of [P10]).

Proof. Let $f \in C_c(T)$ and let $q \in \Gamma$. Then by Theorem 51.B of [H], f is $\sigma(\mathcal{P})$-measurable. Let $\operatorname{supp} f = K \in \mathcal{C}$. Then by Theorem 50.D of [H] there exists $C_0 \in \mathcal{C}_0$ such that $K \subset C_0$. As $N(f) \subset C_0$, for a Borel set B in $\boldsymbol{K}$we have $f^{-1}(B) \cap N(f) \in \sigma(\mathcal{P}) \cap C_0 = \sigma(\mathcal{P} \cap C_0)$ by Theorem 5.E of [H]. As $\mathcal{P} \cap C_0$ is a σ-ring, it follows that f is $\mathcal{P} \cap C_0$-measurable. Hence there exists a sequence (s_n) of $(\mathcal{P} \cap C_0)$-simple functions such that $s_n \to f$ and $|s_n| \nearrow |f|$ uniformly in T. Then for $A \in \sigma(\mathcal{P})$, by Theorem 3.5(i) of [P8] we have

$$|| \int_A |s_n|^p d\mathbf{m} - \int_A |s_k|^p d\mathbf{m}||_q \leq |||s_n|^p - |s_k|^p||_T ||\mathbf{m}||_q(C_0) \to 0$$

as $n, k \to \infty$. As q is arbitrary in Γ and as X is sequentially complete, we conclude that there exists $x_A \in X$ such that $\lim_n \int_A |s_n|^p d\mathbf{m} = x_A$. This holds for each $A \in \sigma(\mathcal{P})$ and consequently, by Definition 12.1' in Remark 12.11 of [P10], $|f|^p$ is $\mathbf{m}$-integrable in T and hence $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$.

Lemma 20.9. Let $\mathcal{S} = \mathcal{B}(T)$ or $\mathcal{B}_c(T)$ or $\mathcal{B}_0(T)$, X be a sequentially complete lcHs and $\mathbf{m} : \mathcal{S} \to X$ be σ-additive. Then $C_0(T) \subset \mathcal{L}_p(\sigma(\mathcal{S}), \mathbf{m}) = \mathcal{L}_p(\mathcal{S}, \mathbf{m})$ for $1 \leq p < \infty$.

Proof. Given $f \in C_0(T)$, f is bounded and by Theorem 51.B of [H] f is $\mathcal{S}$-measurable and hence there exists a sequence (s_n) of $\mathcal{S}$-simple functions such that $s_n \to f$ and $|s_n| \nearrow |f|$ uniformly in T. Then arguing as in the last part of the proof of Lemma 20.8 we conclude that $f \in \mathcal{L}_p(\mathcal{S}, \mathbf{m})$ for $1 \leq p < \infty$.

Theorem 20.10. Let X be a sequentially complete (resp. quasicomplete) lcHs and let $1 \leq p < \infty$. Suppose $\mathbf{m} : \mathcal{P} \to X$ is σ-additive when $\mathcal{P} = \delta(\mathcal{C}_0)$ or $\mathcal{B}_0(T)$; and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $\mathcal{P}$-regular when $\mathcal{P} = \delta(\mathcal{C})$ or $\mathcal{B}_c(T)$ or $\mathcal{B}(T)$. Then $C_c(T)$ is dense in $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ (resp. in $\mathcal{L}_p(\mathbf{m})$) and then, given $f \in \mathcal{L}_p(\mathbf{m})$ (resp. $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$), $q \in \Gamma$ and $\epsilon > 0$, there exists $g_q \in C_c(T)$ such that $(\mathbf{m}_q)^\bullet_p(f - g_q, T) < \epsilon$ (resp. and $||g_q||_T \leq ||f||_T$). If $\mathcal{P} = \mathcal{B}_0(T)$ or $\mathcal{B}_c(T)$ or $\mathcal{B}(T)$, then $C_0(T)$ is also dense in $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ (resp. in $\mathcal{L}_p(\mathbf{m})$).

Proof. By Lemma 20.8, $C_c(T) \subset \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ for $\mathcal{P} = \delta(\mathcal{C}_0)$ or $\delta(\mathcal{C})$ and by Lemma 20.9, $C_c(T) \subset C_0(T) \subset \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ for $\mathcal{P} = \mathcal{B}_0(T)$ or $\mathcal{B}_c(T)$ or $\mathcal{B}(T)$. When X is quasicomplete, $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}) \subset \mathcal{L}_p(\mathbf{m})$. Let $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ (resp. $f \in \mathcal{L}_p(\mathbf{m})$). Let $q \in \Gamma$ and $\epsilon > 0$. Then by Theorem 15.6 of [P10] there exists a $\mathcal{P}$-simple function s such that $(\mathbf{m}_q)^\bullet_p(f - s, T) < \frac{\epsilon}{2}$ and when $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$, by the same theorem we can choose s further to satisfy $|s(t)| \leq |f(t)|$ for t in T. Then by Theorems 20.2(ii) and 20.6 there exists $g_q \in C_c(T)$ such that $||\mathbf{m}||_q(N(g_q - s)) < ((\frac{\epsilon}{2})(\frac{1}{2||s||_T}))^p$ and $||g_q||_T \leq ||s||_T$. Now by Theorem 13.2 and by Proposition 10.14(c) of [P10] we have

$$\begin{aligned}(\mathbf{m}_q)^\bullet_p(s - g_q, T) &= (\mathbf{m}_q)^\bullet_p(s - g_q, N(s - g_q)) \\ &= \sup_{x^* \in U_q^0} \left(\int_{N(s-g_q)} |s - g_q|^p dv(x^*\mathbf{m}) \right)^{\frac{1}{p}} \\ &\leq 2||s||_T(||\mathbf{m}||_q(N(s - g_q)))^{\frac{1}{p}} < \frac{\epsilon}{2}\end{aligned}$$

and hence, by Theorem 5.13(ii) of [P9] we have

$$(\mathbf{m}_q)^\bullet_p(f - g_q, T) \leq (\mathbf{m}_q)^\bullet_p(f - s, T) + (\mathbf{m}_q)^\bullet_p(s - g_q, T) < \epsilon.$$

Moreover, for $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$, $||g_q||_T \leq ||s||_T \leq ||f||_T$. Hence the theorem holds.

Remark 20.11. Restricting the agument in the proof of Theorem 20.10 to real functions, we have similar results for $\mathcal{L}^r_p(\sigma(\mathcal{P}), \mathbf{m})$ and $\mathcal{L}^r_p(\mathbf{m})$ with $C_c(T)$ and $C_0(T)$ being replaced by $C^r_c(T)$ and $C^r_0(T)$, respectively.

Theorem 20.12. Let X be an lcHs and let $\mathbf{m} : \mathcal{B}(T) \to X$ be σ-additive and Borel regular. Then $\mathbf{m}_c = \mathbf{m}|_{\mathcal{B}_c(T)}$ (resp. $\mathbf{m}_0 = \mathbf{m}|_{\mathcal{B}_0(T)}$) is σ-additive and $\mathcal{B}_c(T)$-regular (resp. and Baire regular). Consequently, $\mathbf{m}|_{\delta(\mathcal{C})}$ (resp. $\mathbf{m}|_{\delta(\mathcal{C}_0)}$) is σ-additive and $\delta(\mathcal{C})$-regular (resp. and $\delta(\mathcal{C}_0)$-regular).

Proof. Let $A \in \mathcal{B}_c(T)$. Then there exists a sequence $(K_n) \subset \mathcal{C}$ such that $A \subset \bigcup_1^\infty K_n$. Let $q \in \Gamma$ and $\epsilon > 0$. Then by hypothesis there exists an open set U_n in T such that $A \cap K_n \subset U_n$ with $||\mathbf{m}||_q(U_n \backslash (A \cap K_n)) < \frac{\epsilon}{2^{n+1}}$ for $n \in \mathbb{N}$. By Theorem 50.D of [H] there exists a (σ-Borel) relatively compact open set V_n in T such that $K_n \subset V_n$ so that $A \cap K_n \subset V_n$. Let $W_n = U_n \cap V_n$. Then $W = \bigcup_1^\infty W_n$ is a σ-Borel open set in T and $A \subset W$. By hypothesis, there exists $K \in \mathcal{C}$ such that $K \subset A$ with $||\mathbf{m}||_q(A \backslash K) < \frac{\epsilon}{2}$. Then $K \subset A \subset W$ and $||\mathbf{m}||_q(W \backslash K) < \epsilon$. In fact, $W \backslash A \subset \bigcup_1^\infty (W_n \backslash (A \cap V_n)) \subset \bigcup_1^\infty (W_n \backslash (A \cap K_n)) \subset \bigcup_1^\infty (U_n \backslash (A \cap K_n))$. As $||\mathbf{m}||_q$ is σ-subadditive on $\mathcal{B}(T)$, we have $||\mathbf{m}||_q(W \backslash A) < \frac{\epsilon}{2}$. Consequently, $||\mathbf{m}||_q(W \backslash K) \le ||\mathbf{m}||_q(W \backslash A) + ||\mathbf{m}||_q(A \backslash K) < \epsilon$ and hence $\mathbf{m}_c$ is $\mathcal{B}_c(T)$-regular. Then the other results hold by Lemmas 20.3 and 20.5.

21. LUSIN MEASURABILITY OF FUNCTIONS AND SETS

If X is an lcHs and $\mathbf{m} : \mathcal{P} \to X$ is σ-additive, let us recall from Definition 10.6 of [P10] that for a set A in T, χ_A is $\mathbf{m}$-measurable if $A \in \widetilde{\sigma(\mathcal{P})_q}$, the generalized Lebesgue completion of $\sigma(\mathcal{P})$ with respect to $||\mathbf{m}||_q$ for each $q \in \Gamma$. In that case, we say that A is $\mathbf{m}$-measurable. When $\mathcal{P} = \mathcal{B}(T)$ (resp. $\delta(\mathcal{C})$) and $\mathbf{m}$ is further $\mathcal{P}$-regular, we introduce the concept of Lusin $\mathbf{m}$-measurability and study the inter-relations between the concepts of $\mathbf{m}$-measurability and Lusin $\mathbf{m}$-measurability in Theorems 21.5 and 21.6. The latter theorems play a key role in Section 22.

Theorem 21.1. Let X be an lcHs and $\mathbf{m} : \mathcal{B}(T) \to X$ be σ-additive and Borel regular. For a set A in T the following statements are equivalent:

(i) A is $\mathbf{m}$-measurable.

(ii) Given $q \in \Gamma$ and $\epsilon > 0$, there exist $K_q \in \mathcal{C}$ and an open set U_q in T such that $K_q \subset A \subset U_q$ and $||\mathbf{m}||_q(U_q \backslash K_q) < \epsilon$.

(iii) Given $q \in \Gamma$, there exist a G_δ G_q and an F_σ F_q in T such that $F_q \subset A \subset G_q$ with $||\mathbf{m}||_q(G_q \backslash F_q) = 0$.

(iv) Given $q \in \Gamma$, there exist a disjoint sequence $(K_n^{(q)})_1^\infty \subset \mathcal{C}$ and a G_δ G_q in T such that $F_q = \bigcup_{n=1}^\infty K_n^{(q)} \subset A \subset G_q$ with $||\mathbf{m}||_q(G_q \backslash F_q) = 0$.

(v) For each $q \in \Gamma$, $A \cap K \in \widetilde{\mathcal{B}(T)_q}$ for each $K \in \mathcal{C}$.

(vi) For each $q \in \Gamma$, $A \cap U \in \widetilde{\mathcal{B}(T)_q}$ for each open set U in T.

Proof. Without loss of generality we shall assume X to be a normed space.

(i)⇒(ii) By the Borel regularity of $\mathbf{m}$ and by the fact that the $\mathbf{m}$-measurable set A is of the form $A = B \cup N$, $N \subset M \in \mathcal{B}(T)$, $B \in \mathcal{B}(T)$ and $||\mathbf{m}||(M) = 0$, (i)⇒(ii).

(ii)⇒(iii) By (ii), for $\epsilon = \frac{1}{n}$, $n \in \mathbb{N}$, there exist a compact K_n and an open set U_n in T such that $K_n \subset A \subset U_n$ with $||\mathbf{m}||(U_n \backslash K_n) < \frac{1}{n}$. Let $G = \bigcap_1^\infty U_n$ and $F = \bigcup_1^\infty K_n$. Then G is a

G_δ, F is an F_σ, $F \subset A \subset G$ and $||\mathbf{m}||(G\backslash F) \leq ||\mathbf{m}||(U_n\backslash K_n) < \frac{1}{n}$ for each $n \in I\!N$ and hence $||\mathbf{m}||(G\backslash F) = 0$.

(iii)⇒(i) Let $F \subset A \subset G$. F an F_σ, G a G_δ with $||\mathbf{m}||(G\backslash F) = 0$. Then $A = F \cup (A\backslash F)$, $A\backslash F \subset G\backslash F$, $(G\backslash F) \in \mathcal{B}(T)$ and $||\mathbf{m}||(G\backslash F) = 0$. Hence $A \in \widetilde{\mathcal{B}(T)}$ and hence (i) holds.

Thus (i), (ii) and (iii) are equivalent.

(i)⇔(ii)⇒(iv) By hypothesis, there exists $K_1 \in \mathcal{C}$ such that $K_1 \subset A$ and $||\mathbf{m}||(A\backslash K_1) < 1$. Since $A\backslash K_1 \in \widetilde{\mathcal{B}(T)}$, by (ii) there exists $K_2 \in \mathcal{C}$ such that $K_2 \subset A\backslash K_1$ and $||\mathbf{m}||(A\backslash(K_1 \cup K_2)) < \frac{1}{2}$. Proceeding step by step, in the n^{th} step we would have chosen mutually disjoint compact sets $(K_1)_1^n$ such that $\bigcup_1^n K_i \subset A$ with $||\mathbf{m}||(A\backslash \bigcup_1^n K_i) < \frac{1}{n}$. Then $F = \bigcup_1^\infty K_i \subset A$ and $||\mathbf{m}||(A\backslash F) = 0$. Moreover, by (iii)⇔(i) there exists a G_δ G such that $A \subset G$ and $||\mathbf{m}||(G\backslash A) = 0$. Hence (iv) holds.

(iv)⇒(i) Let F and G be as in the hypothesis. Then $F, G \in \mathcal{B}(T)$ and $||\mathbf{m}||(G\backslash F) = 0$. Since $A = F \cup (A\backslash F)$, it follows that $A \in \widetilde{\mathcal{B}(T)}$ and hence (i) holds.

Thus (i),(ii), (iii) and (iv) are equivalent.

(i)⇒(v) obviously.

(v)⇒(vi) Let U be an open set in T. Then $U \in \mathcal{B}(T)$ and hence (iv) holds. Thus there exists $(K_n)_1^\infty \subset \mathcal{C}$ such that $\bigcup_1^\infty K_n \subset U$ and $N = U\backslash \bigcup_1^\infty K_n$ is $\mathbf{m}$-null. Then by (v), $A \cap U = \bigcup_1^\infty (A \cap K_n) \cup (A \cap N) \in \widetilde{\mathcal{B}(T)}$. Hence (vi) holds.

(vi)⇒(i) by taking $U = T$.

Hence (i)⇒(v)⇒(vi)⇒(i).

This completes the proof of the theorem.

Theorem 21.2. Let X be an lcHs and let $\mathbf{n} : \delta(\mathcal{C}) \to X$ be σ-additive and $\delta(\mathcal{C})$-regular. For a set A in T the following statements are equivalent:

(i) A is $\mathbf{n}$-measurable.

(ii) Given $q \in \Gamma$, there exist a σ-compact F_q and a G_δ $G_q \in \mathcal{B}_c(T)$ such that $F_q \subset A \subset G_q$ with $||\mathbf{n}||_q(G_q\backslash F_q) = 0$.

(iii) A is σ-bounded and $A \cap K \in \widetilde{\mathcal{B}_c(T)}_q$ for each $K \in \mathcal{C}$ and for each $q \in \Gamma$.

(iv) A is σ-bounded and $A \cap U \in \widetilde{\mathcal{B}_c(T)}_q$ for each open set U in T and for each $q \in \Gamma$.

Proof. Without loss of generality we shall assume X to be a normed space.

(i)$\Rightarrow$(ii) By hypothsis, A is of the form $A = B \cup N$, $B \in \mathcal{B}_c(T)$, $N \subset M \in \mathcal{B}_c(T)$ and $||\mathbf{n}||(M) = 0$. As $(B \cup M) \in \mathcal{B}_c(T)$, there exists $(K_n)_1^\infty \subset \mathcal{C}$ such that $(B \cup M) \subset \bigcup_1^\infty K_n$ so that $B \cup M = \bigcup_1^\infty ((B \cup M) \cap K_n)$. Then by Lemma 18.2 of [P11], $((B \cup M) \cap K_n)_1^\infty \subset \delta(\mathcal{C})$. As $\mathbf{n}$ is $\delta(\mathcal{C})$-regular, given $\epsilon > 0$, there exists an open set V_n in T such that $V_n \in \delta(\mathcal{C})$ and such that $(B \cup M) \cap K_n \subset V_n$ with $||\mathbf{n}||(V_n \backslash ((B \cup M) \cap K_n)) < \frac{\epsilon}{2^n}$. Then $U = \bigcup_1^\infty V_n$ is an open set in T belonging to $\mathcal{B}_c(T)$, $(B \cup M) \subset U$ and $||\mathbf{n}||(U \backslash (B \cup M)) < \epsilon$. Hence $A \subset U$ and $||\mathbf{n}||(U \backslash A) < \epsilon$. By taking $\epsilon = \frac{1}{n}$, $n \in I\!N$ we obtain open sets $U_n \in \mathcal{B}_c(T)$ such that $A \subset U_n$ and $||\mathbf{n}||(U_n \backslash A) < \frac{1}{n}$. Then $G = \bigcap_1^\infty U_n$ is a G_δ, $G \in \mathcal{B}_c(T)$, $A \subset G$ and $||\mathbf{n}||(G \backslash A) = 0$. As $\mathbf{n}$ is $\delta(\mathcal{C})$-regular, given $k \in I\!N$, there exists $C_n^{(k)} \in \mathcal{C}$ such that $C_n^{(k)} \subset B \cap K_n \in \delta(\mathcal{C})$ with $||\mathbf{n}||(B \cap K_n) \backslash C_n^{(k)}) < \frac{1}{k} \cdot \frac{1}{2^n}$ for $n \in I\!N$. If $F_k = \bigcup_{n=1}^\infty C_n^{(k)}$, then $F = \bigcup_1^\infty F_k$ is σ-compact, $F \subset B \subset A$ and $||\mathbf{n}||(A \backslash F) = ||\mathbf{n}||(B \backslash F) = 0$. Then $F \subset A \subset G$ with $||\mathbf{n}||(G \backslash F) = 0$ and hence (ii) holds.

(ii)$\Rightarrow$(i) since $A = F \cup (A \backslash F)$, $F \in \mathcal{B}_c(T)$, $A \backslash F \subset (G \backslash F) \in \mathcal{B}_c(T)$ and $||\mathbf{n}||(G \backslash F) = 0$.

Thus (i)$\Leftrightarrow$(ii).

(i)$\Rightarrow$(iii) and (iv) Take B, N and M as in the proof of '(i)$\Rightarrow$(ii)'. Clearly A is σ-bounded. For $K \in \mathcal{C}$, $B \cap K \in \mathcal{B}_c(T)$ and $N \cap K$ is $\mathbf{n}$-null so that $N \cap K \in \widetilde{\mathcal{B}_c(T)}$. Hence $A \cap K \in \widetilde{\mathcal{B}_c(T)}$. For an open set U in T, $U \cap B$ is σ-bounded and hence $U \cap B \in \mathcal{B}_c(T)$ and $U \cap N$ is $\mathbf{n}$-null so that $U \cap N \in \widetilde{\mathcal{B}_c(T)}$. Hnce $U \cap A \in \widetilde{\mathcal{B}_c(T)}$.

(iii)$\Rightarrow$(i) As A is σ-bounded, there exists $(K_n)_1^\infty \subset \mathcal{C}$ such that $A \subset \bigcup_1^\infty K_n$ so that by hypothesis, $A = \bigcup_1^\infty (A \cap K_n) \in \widetilde{\mathcal{B}_c(T)}$.

(iv)$\Rightarrow$(i) As A is σ-bounded, by Theorem 50.D of [H] there exists relatively compact open sets $(U_n)_1^\infty$ in T such that $A \subset \bigcup_1^\infty U_n$. Then $A = \bigcup_1^\infty (A \cap U_n) \in \widetilde{\mathcal{B}_c(T)}$ by (iv).

Hence (i)$\leftrightarrow$(iii)$\leftrightarrow$(iv).

This completes the proof of the theorem.

Definition 21.3. Let X be an lcHs and let $\mathbf{m} : \mathcal{B}(T) \to X$ (resp. $\mathbf{n} : \delta(\mathcal{C}) \to X$) be σ-additive and $\mathcal{B}(T)$-regular (resp. and $\delta(\mathcal{C})$-regular). Then a functon $f : T \to I\!\!K$ is said to be Lusin $\mathbf{m}$-measurable (resp. Lusin $\mathbf{n}$-measurable) if, given $q \in \Gamma$, $\epsilon > 0$ and $K \in \mathcal{C}$, there exists a $K_0^{(q)} \subset K$ such that $f|_{K_0^{(q)}}$ is continuous and $||\mathbf{m}||_q(K \backslash K_0^{(q)}) < \epsilon$ (resp. and $||\mathbf{n}||_q(K \backslash K_0^{(q)}) < \epsilon$).

Theorem 21.4. Let X, $\mathbf{m}$, $\mathbf{n}$ and f be as in Definition 21.3. Then f is Lusin $\mathbf{m}$-measurable (resp. Lusin $\mathbf{n}$-measurable) if and only if, given $q \in \Gamma$ and $K \in \mathcal{C}$, there exist an $\mathbf{m}_q$-null set (resp. $\mathbf{n}_q$-null set) $N_q \subset K$ and a countable disjoint family $(K_i^{(q)})_1^\infty \subset \mathcal{C}$ such that $K = \bigcup_{i=1}^\infty K_i^{(q)} \cup N_q$ and $f|_{K_i^{(q)}}$ is continuous for each $i \in \mathbb{N}$.

Proof. Without loss of generality we shall assume X to be a normed space. Let $\mathcal{P} = \mathcal{B}(T)$ and $\boldsymbol{\omega} = \mathbf{m}$ or $\mathcal{P} = \delta(\mathcal{C})$ and $\boldsymbol{\omega} = \mathbf{n}$. Let f be Lusin $\boldsymbol{\omega}$-measurable and $K \in \mathcal{C}$. Then by Definition 21.3 there exists $K_1 \in \mathcal{C}$ such that $K_1 \subset K$, $f|_{K_1}$ is continuous and $||\boldsymbol{\omega}||(K \backslash K_1) < 1$. Let $n > 1$ and suppose we have chosen $(K_1)_1^n \subset \mathcal{C}$ mutually disjoint such that $\bigcup_1^n K_i \subset K$, $f|_{K_i}$ is continuous for $1 \le i \le n$ and $||\boldsymbol{\omega}||(K \backslash \bigcup_1^n K_i) < \frac{1}{n}$. As $K \backslash \bigcup_1^n K_i \in \delta(\mathcal{C})$, by the regularity of $\boldsymbol{\omega}$ there exists a compact $C \subset K \backslash \bigcup_1^n K_i$ such that $||\boldsymbol{\omega}||(K \backslash (\bigcup_1^n K_i \cup C)) < \frac{1}{2(n+1)}$. By hypothesis there exists a compact $K_{n+1} \subset C$ such that $f|_{K_{n+1}}$ is continuous and $||\boldsymbol{\omega}||(C \backslash K_{n+1}) < \frac{1}{2(n+1)}$. Then $(K_i)_1^{n+1} \subset \mathcal{C}$ are mutually disjoint, $\bigcup_1^{n+1} K_i \subset K$ and $||\boldsymbol{\omega}||(K \backslash \bigcup_1^{n+1} K_i) < \frac{1}{n+1}$. Therefore, by induction there exists a disjoint sequence $(K_i)_1^\infty \subset \mathcal{C}$ such that $f|_{K_i}$ is continuous for all i and $||\boldsymbol{\omega}||(K \backslash \bigcup_1^n K_i) < \frac{1}{n}$ for all n. Then $N = K \backslash \bigcup_1^\infty K_i$ is $\boldsymbol{\omega}$-null and $f|_{K_i}$ is continuous for all i.

Conversely, let $K \in \mathcal{C}$ and suppose $K = \bigcup_1^\infty K_i \cup N$, where $(K_i)_1^\infty \subset \mathcal{C}$, $K_i \cap K_j = \emptyset$ for $i \neq j$, $f|_{K_i}$ is continuous for each i and $||\boldsymbol{\omega}||(N) = 0$. Let $\epsilon > 0$. As $K \backslash \bigcup_1^n K_i \in \mathcal{P}$ for all n, $K \backslash \bigcup_1^n K_i \searrow N \in \delta(\mathcal{C})$ and as $||\boldsymbol{\omega}||$ is continuous on $\mathcal{P}$ by Proposition 2.1 of [P8], there exists n_0 such that $||\boldsymbol{\omega}||(K \backslash \bigcup_1^{n_0} K_i) < \epsilon$. Clearly, $K_0 = \bigcup_1^{n_0} K_i \in \mathcal{C}$, $K_0 \subset K$ and $f|_{K_0}$ is continuous since K_i are mutually disjoint. Hence f is Lusin $\boldsymbol{\omega}$-measurable.

Theorem 21.5. Let X be an lcHs, $\mathbf{m} : \mathcal{B}(T) \to X$ be σ-additive and Borel regular and $f : T \to \mathbb{K}$. Then f is Lusin $\mathbf{m}$-measurable if and only if it is $\mathbf{m}$-measurable.

Proof. Without loss of generality we shall assume X to be a normed space. Let f be $\mathbf{m}$-measurable, $K \in \mathcal{C}$ and $\epsilon > 0$. Then $f\chi_K$ is $\mathbf{m}$-measurable and hence by Proposition 2.10 of [P8] there exists $N \in \mathcal{B}(T)$ with $||\mathbf{m}||(N) = 0$ such that $h = f\chi_{K \backslash N}$ is $\mathcal{B}(T)$-measurable. Then by Theorem 20.2(i) there exists $g \in C_c(T)$ such that $||\mathbf{m}||(N(h-g)) < \frac{\epsilon}{2}$. Let $A = N(h-g)$. Then $A \in \mathcal{B}(T)$ and hence by the Borel regularity of $\mathbf{m}$ there exists a compact $K_0 \subset K \backslash A$ such that $||\mathbf{m}||(K \backslash A \backslash K_0) < \frac{\epsilon}{2}$. Then $h|_{K_0} = f|_{K_0} = g|_{K_0}$ is continuous and $||\mathbf{m}||(K \backslash K_0) < \epsilon$. Hence f is Lusin $\mathbf{m}$-measurable.

Conversely, let f be Lusin $\mathbf{m}$-measurable. Given $K \in \mathcal{C}$, by Theorem 21.4 there exist a disjoint sequence $(K_i)_1^\infty \subset \mathcal{C}$ and an $\mathbf{m}$-null set N disjoint with $\bigcup_1^\infty K_i$ such that $K = \bigcup_1^\infty K_i \cup N$ and such that $f|_{K_i}$ is continuous for each i. Let U be an open set in $\mathbb{K}$. Then $f^{-1}(U) \cap K = \bigcup_1^\infty (f^{-1}(U) \cap K_i) \cup (f^{-1} \cap N)$. As $f|_{K_i}$ is continuous, there exists an open set V_i in T such that $f^{-1}(U) \cap K_i = V_i \cap K_i$ and hence $f^{-1}(U) \cap K = \bigcup_1^\infty (V_i \cap K_i) \cup (f^{-1}(U) \cap N) \in \widetilde{\mathcal{B}(T)}$. Then by Theorem 21.1(v), f is $\mathbf{m}$-measurable.

Theorem 21.6. Let X be an lcHs, $\mathbf{n} : \delta(\mathcal{C}) \to X$ be σ-additive and $\delta(\mathcal{C})$-regular and $f : T \to \mathbb{K}$. Then f is $\mathbf{n}$-measurable if and only if $N(f)$ is σ-bounded and f is Lusin $\mathbf{n}$-measurable.

Proof. Without loss of generality we shall assume X to be a normed space. Let f be $\mathbf{n}$-measurable. Then $N(f) \in \widetilde{\mathcal{B}_c(T)}$ and hence $N(f)$ is σ-bounded. Let $K \in \mathcal{C}$ and $\epsilon > 0$. Then by Proposition 2.10 of [P8] there exists $N \in \mathcal{B}_c(T)$ with $||\mathbf{n}||(N) = 0$ such that $f\chi_{T\backslash N}$ is $\mathcal{B}_c(T)$-measurable. Then $f\chi_{K\backslash N}$ is $\mathcal{B}_c(T)$-measurable. Hence by Theorem 20.6 there exists $g \in C_c(T)$ such that $||\mathbf{n}||(N(f\chi_{K\backslash N} - g)) < \frac{\epsilon}{2}$. Let $A = N(g - f\chi_{K\backslash N})$. Then $A \in \mathcal{B}_c(T)$ and hence $K\backslash A \in \delta(\mathcal{C})$ by Lemma 18.2 of [P11]. Then by the $\delta(\mathcal{C})$-regularity of $\mathbf{n}$ there exists a compact $K_0 \subset K\backslash A$ such that $||\mathbf{n}||(K\backslash A\backslash K_0) < \frac{\epsilon}{2}$. Then $f|_{K_0} = g|_{K_0}$ is continuous and $||\mathbf{n}||(K\backslash K_0) < \epsilon$. Hence f is Lusin $\mathbf{n}$-measurable.

Conversely, let f be Lusin $\mathbf{n}$-measurable and let $N(f)$ be σ-bounded. Let $K \in \mathcal{C}$. Then by Theorem 21.4 there exist a disjoint countable family $(K_i)_1^\infty \subset \mathcal{C}$ and an $\mathbf{n}$-null set N disjoint with $\bigcup_1^\infty K_i$ such that $K = \bigcup_1^\infty K_i \cup N$ and $f|_{K_i}$ is continuous for each i. Let U be an open set in $\boldsymbol{K}$ If $f_i = f|_{K_i}$, then by the continuity of f_i we have $f^{-1}(U\backslash\{0\}) \cap K_i = f_i^{-1}(U\backslash\{0\}) \in \mathcal{B}(K_i)$ and hence $N(f) \cap f^{-1}(U) \cap K = \bigcup_1^\infty (f^{-1}(U\backslash\{0\}) \cap K_n) \cup (N \cap f^{-1}(U\backslash\{0\})) \in \widetilde{\mathcal{B}_c(T)}$. As $N(f)$ is σ-bounded by hypothesis, it follows by Theorem 21.2(v) that $N(f) \cap f^{-1}(U) \in \widetilde{\mathcal{B}_c(T)}$ and hence f is $\mathbf{n}$-measurable.

Corollary 21.7. Let X be an lcHs and let $\mathbf{m} : \mathcal{B}(T) \to X$ (resp. $\mathbf{n} : \delta(\mathcal{C}) \to X$) be σ-additive and Borel regular (resp. and $\delta(\mathcal{C})$-regular). Then a Borel measurable scalar function f on T is Lusin $\mathbf{m}$-measurable (resp. Lusin $\mathbf{n}$-measurable).

Proof. Let f be Borel measurable. Then f is $\mathbf{m}$-measurable and hence is Lusin $\mathbf{m}$-measurable by Theorem 21.5. Let $K \in \mathcal{C}$. Then by Lemma 18.2 of [P11], $f\chi_K$ is $\mathcal{B}_c(T)$-measurable and hence $\mathbf{n}$-measurable. Clearly, $N(f\chi_K)$ is σ-bounded. Hence by Theorem 21.6, $f\chi_K$ is Lusin $\mathbf{n}$-measurable. As K is arbitrary in $\mathcal{C}$, it follows that f is Lusin $\mathbf{n}$-measurable.

Definition 21.8. Let X be an lcHs and let $\mathbf{m} : \mathcal{B}(T) \to X$ (resp. $\mathbf{n} : \delta(\mathcal{C}) \to X$) be σ-additive and Borel regular (resp. and $\delta(\mathcal{C})$-regular). Then a set A in T is said to be Lusin $\mathbf{m}$-measurable (resp. Lusin $\mathbf{n}$-measurable) if χ_A is so.

The following theorem is immediate from Definition 21.8 and Theorem 21.4.

Theorem 21.9. Let X, $\mathbf{m}$ and $\mathbf{n}$ be as in Definition 21.8. Let $A \subset T$. Then A is Lusin $\mathbf{m}$-measurable (resp. Lusin $\mathbf{n}$-measurable) if for each $q \in \Gamma$ and $K \in \mathcal{C}$, there exist a disjoint sequence $(K_i^{(q)})_1^\infty \subset \mathcal{C}$ and an $\mathbf{m}_q$-null set (resp. and an $\mathbf{n}_q$-null set) N_q disjoint with $\bigcup_1^\infty K_i$ such that $K = \bigcup_1^\infty K_i^{(q)} \cup N_q$ and such that, for each i, $K_i^{(q)} \subset A$ or $K_i^{(q)} \subset T\backslash A$.

Using Theorem 21.1(iv) and the Borel regularity of $\mathbf{m}$, the proof of Proposition 4, no.2, §5, Ch. IV of [B] can be adapted to prove the following

Theorem 21.10 (Localization principle). Let X be an lcHs and let $\mathbf{m} : \mathcal{B}(T) \to X$ be

σ-additive and Borel regular. Let $f : T \to \boldsymbol{K}$ and suppose for each $t \in T$ and $q \in \Gamma$, there exist an open neighborhood $V_t^{(q)}$ of t and a Lusin $\mathbf{m}_q$-measurable scalar function $g_t^{(q)}$ such that $f(t') = g_t^{(q)}(t')$ $\mathbf{m}_q$-a.e. in $V_t^{(q)}$. Then f is Lusin $\mathbf{m}$-measurable.

As in the classical case of [B], the above theorem motivates the following

Definition 21.11. Let X and $\mathbf{m}$ be as in Theorem 21.10. A set A in T is said to be locally $\mathbf{m}$-null (briefly, loc. $\mathbf{m}$-null) if, for each $t \in T$, there exists an open neighborhood V_t of t such that $A \cap V_t$ is $\mathbf{m}$-null. (See Definition 10.3 of [P10].)

The proof of the following theorem is similar to those on pp. 172-173 of [B] and is based on Theorems 21.1(iv), 21.5 and 21.10 and hence is omitted.

Theorem 21.12. Let X be an lcHs and let $\mathbf{m} : \mathcal{B}(T) \to X$ be σ-additive and Borel regular. Then:

(i) Locally $\mathbf{m}$-null sets are $\mathbf{m}$-measurable.

(ii) If A is loc. $\mathbf{m}$-null, then all the subsets of A are also loc. $\mathbf{m}$-null.

(iii) A is loc. $\mathbf{m}$-null if and only if $A \cap K$ is $\mathbf{m}$-null for each $K \in \mathcal{C}$.

(iv) If A_i, $i \in \mathbb{N}$, are locally $\mathbf{m}$-null, then $\bigcup_1^\infty A_i$ is also loc. $\mathbf{m}$-null.

(v) A is loc. $\mathbf{m}$-null if and only if A is $\mathbf{m}$-null. (Use (i) and Theorem 21.1(iv).)

(vi) $f : T \to \boldsymbol{K}$ and $N = \{t \in T : f \text{ is discontinuous in } t\}$ is loc. $\mathbf{m}$-null, then f is $\mathbf{m}$-measurable.

22. THEOREMS OF INTEGRABILITY CRITERIA

The aim of the present section is to improve Theorem 4.2 of [P8] and Theorem 12.2 of [P10] for $\delta(\mathcal{C})$-regular σ-additive vector measures on $\delta(\mathcal{C})$. The said improvement of Theorem 4.2 of [P8] is given in the last part of Therem 22.4 which gives much stronger results and Theorem 22.5 improves Theorem 12.2 of [P10]. We also generlize Theorem 22.4 to complete lcHs valued vector measures. The proofs of Lemmas 3.10 and 3.14, Propositions 2.17, 2.20 and 3.7 and Theorems 3.5, 3.13 and 3.20 of [T] are adapted here in the set-up of vector measures.

Recall from Notation 19.2 of [P11] that $\mathcal{V}$ denotes the family of relatively compact open sets in T.

Lemma 22.1. Let X be a Banach space and let H be a norm determining set in X^*. Let $\mathcal{P}$ be a δ-ring of subsets of a set $\Omega(\neq \emptyset)$ and let $\mathbf{m} : \mathcal{P} \to X$ be additive. Then:

(i) $||\mathbf{m}||(A) = \sup_{x^* \in H} v(x^* \circ \mathbf{m})(A)$, $A \in \sigma(\mathcal{P})$.

(ii) Suppose $\mathbf{m}$ is σ-additive and $f : \Omega \to \boldsymbol{K}$ is $\mathbf{m}$-measurable and $(x^* \circ \mathbf{m})$-integrable for each $x^* \in H$. If, for each $\epsilon > 0$, there exists $g_\epsilon \in \mathcal{L}_1(\mathbf{m})$ such that $\sup_{x^* \in H} \int_T |f - g_\epsilon| dv(x^* \circ \mathbf{m}) < \epsilon$, then $f \in \mathcal{L}_1(\mathbf{m})$.

Proof. (i) This is proved by an argument similar to that in the proof of Proposition 10.12(iii) of [P10].

(ii) Let $\nu_{x^*}(\cdot) = \int_{(\cdot)} f d(x^* \circ \mathbf{m})$, $x^* \in H$. Then by hypothesis and by Proposition 5, §8 of [Din], ν_{x^*} is σ-additive on $\sigma(\mathcal{P})$ and by Proposition 2.11 of [P8], $v(\nu_{x^*})(A) = \int_A |f| dv(x^* \circ \mathbf{m})$ for $A \in \sigma(\mathcal{P})$. Let

$$\eta(f) = \sup_{x^* \in H} \int_T |f| dv(x^* \circ \mathbf{m}).$$

If $f \in \mathcal{L}_1(\mathbf{m})$, then by (ii) and (iii) of Theorem 3.5 and by Remark 4.3 of [P8], $\boldsymbol{\gamma}(\cdot) = \int_{(\cdot)} f d\mathbf{m}$ is σ-additive on $\sigma(\mathcal{P})$ and $||\boldsymbol{\gamma}||(T) = \sup_{|x^*| \le 1} \int_T |f| dv(x^* \circ \mathbf{m})$. Consequently, by (i) above and by Theorem 5.3 of [P9] we have

$$\begin{aligned} \mathbf{m}_1^\bullet(f, T) &= \sup_{|x^*| \le 1} \int_T |f| dv(x^* \circ \mathbf{m}) = ||\boldsymbol{\gamma}||(T) \\ &= \sup_{x^* \in H} v(x^* \circ \boldsymbol{\gamma})(T) = \sup_{x^* \in H} \int_T |f| dv(x^* \circ \mathbf{m}) = \eta(f). \qquad (22.1.1) \end{aligned}$$

Let $\Sigma = \{f : T \to \boldsymbol{K},\ f\ \mathbf{m}\text{-measurable and } (x^* \circ \mathbf{m})\text{-integrable for each } x^* \in H \text{ with } \eta(f) < \infty\}$. For $f \in \mathcal{L}_1\mathcal{M}(\mathbf{m})$ (see Definition 5.9 of [P9]), we have $\eta(f) = \sup_{x^* \in H} \int_T |f| dv(x^* \circ \mathbf{m}) \le \mathbf{m}_1^\bullet(f, T) < \infty$ and hence $\mathcal{L}_1\mathcal{M}(\mathbf{m}) \subset \Sigma$. Clearly, η is a seminorm on Σ.

Claim 1. $\mathcal{L}_1(\mathbf{m})$ is closed in (Σ, η).

In fact, let $(f_n)_1^\infty \subset \mathcal{L}_1(\mathbf{m})$ and let $f \in \Sigma$ such that $\eta(f_n - f) \to 0$. Then by (22.1.1), $(f_n)_1^\infty$ is Cauchy in $\mathcal{L}_1(\mathbf{m})$. Hence by Theorem 6.8 of [P9], there exists $g \in \mathcal{L}_1(\mathbf{m})$ such that $\lim_n \mathbf{m}_1^\bullet(f_n - g, T) = 0$. Since $H \subset \{x^* \in X^* : |x^*| \le 1\}$ by Lemma 18.13 of [P11], $\eta(f_n - g) \le \mathbf{m}_1^\bullet(f_n - g, T) \to 0$ as $n \to \infty$. Then $\eta(f - g) \le \eta(f - f_n) + \eta(f_n - g) \to 0$ and hence $\eta(f - g) = 0$. Clearly, $f - g$ is $\mathbf{m}$-measurable and hence $N(f - g) = B \cup N$, where $B \in \sigma(\mathcal{P})$ and $N \subset M \in \sigma(\mathcal{P})$ with $||\mathbf{m}||(M) = 0$. Then by (i) or by the fact that $H \subset \{x^* : |x^*| \le 1\}$, $v(x^* \circ \mathbf{m})(M) = 0$ for $x^* \in H$. Now $\sup_{x^* \in H} \int_{B \cup M} |f - g| dv(x^* \circ \mathbf{m}) = \sup_{x^* \in H} \int_B |f - g| dv(x^* \circ \mathbf{m}) \le \eta(f - g) = 0$ and hence $v(x^* \circ \mathbf{m})(B) = 0$ for $x^* \in H$. Then by (i), $||\mathbf{m}||(B) = \sup_{x^* \in H} v(x^* \circ \mathbf{m})(B) = 0$ so that $||\mathbf{m}||(N(f - g)) = 0$. Therefore, $f = g$ $\mathbf{m}$-a.e. in T and hence $f \in \mathcal{L}_1(\mathbf{m})$. Thus the claim holds.

By hypothesis and by (22.1.1) we have

$$\begin{aligned}\sup_{x^*\in H}\int_T |f|dv(x^*\circ \mathbf{m}) &\leq \sup_{x^*\in H}\int_T |g_\epsilon|dv(x^*\circ \mathbf{m}) + \sup_{x^*\in H}\int_T |f-g_\epsilon|dv(x^*\circ \mathbf{m})\\ &< \mathbf{m}_1^\bullet(g_\epsilon, T)+\epsilon<\infty\end{aligned}$$

and hence $f\in\Sigma$. Moreover, the hypothesis in (ii) implies that f belong to the η-closure of $\mathcal{L}_1(\mathbf{m})$ in Σ. Then by Claim 1, $f\in\mathcal{L}_1(\mathbf{m})$.

Lemma 22.2. Let X and H be as in Lemma 22.1. Let $\mathbf{m}:\delta(\mathcal{C})\to X$ (resp. $\mathbf{m}:\mathcal{B}(T)\to X$) be σ-additive. Let $V\in\mathcal{V}$. Then there exist a sequence (x_n^*) in H and a sequence (c_n) of positive numbers such that

$$\lim_{\lambda(A)\to 0}||\mathbf{m}||(A)=0$$

for $A\in\mathcal{B}(V)$ (resp. for $A\in\mathcal{B}(T)$), where

$$\lambda=\sum_1^\infty c_n v(x_n^*\circ\mathbf{m})$$

is σ-additive and finite on $\mathcal{B}(V)$ (resp. on $\mathcal{B}(T)$). (Note that in the case of $\mathbf{m}$-defined on $\delta(\mathcal{C})$, (x_n^*) and λ depend on V). Consequently, $A\in\mathcal{B}(V)$ (resp. $A\in\mathcal{B}(T)$) is $\mathbf{m}$-null if and only if A is $(x^*\circ\mathbf{m})$-null for each $x^*\in H$. If $\mathbf{m}$ is further $\delta(\mathcal{C})$-regular (resp. $\mathcal{B}(T)$-regular) and if $f:T\to\boldsymbol{K}$ is $(x^*\circ\mathbf{m})$-measurable for each $x^*\in H$, then f is Lusin $\mathbf{m}$-measurable as well as $\mathbf{m}$-measurable.

Proof. Let $V\in\mathcal{V}$. As $\mathbf{m}$ is σ-additive on $\delta(\mathcal{C})$ (resp. on $\mathcal{B}(T)$) and as H is norm bounded by Lemma 18.13 of [P11], $\{x^*\circ\mathbf{m}:x^*\in H\}$ is bounded and uniformly σ-additive on $\mathcal{B}(V)$ (resp. on $\mathcal{B}(T)$) and hence by the proof of Theorem IV.9.2 and by Theorem IV.9.1 of [DS], there exist $(x_n^*)_1^\infty\subset H$ and $c_n>0$, $n\in\mathbb{N}$, such that $\lambda=\sum_1^\infty c_n v(x_n^*\circ\mathbf{m})$ is σ-additive and finite on $\mathcal{B}(V)$ (resp. on $\mathcal{B}(T)$) and satisfies

$$\lim_{\lambda(A)\to 0}\sup_{x^*\in H} v(x^*\circ\mathbf{m})(A)=0$$

for $A\in\mathcal{B}(V)$ (resp. $A\in\mathcal{B}(T)$). Then by Lemma 22.1(i),

$$\lim_{\lambda(A)\to 0}||\mathbf{m}||(A)=0 \text{ for } A\in\mathcal{B}(V) \text{ (resp. } A\in\mathcal{B}(T)). \quad (22.2.1)$$

If $A\in\mathcal{B}(V)$ (resp. $A\in\mathcal{B}(T)$) is $x^*\circ\mathbf{m}$-null for each $x^*\in H$, then $\lambda(A)=0$ and hence $||\mathbf{m}||(A)=0$ so that A is $\mathbf{m}$-null. The converse is trivial.

Now let us assume that $\mathbf{m}$ is further $\delta(\mathcal{C})$-regular (resp. $\mathcal{B}(T)$-regular). Let $K\in\mathcal{C}$ and let $\epsilon>0$. Choose $V\in\mathcal{V}$ such that $K\subset V$. Choose $(x_n^*)_1^\infty\subset H$ and $c_n>0$, $n\in\mathbb{N}$, and λ as above. By (22.2.1), there exists $\delta>0$ such that $||\mathbf{m}||(A)<\epsilon$ whenever $\lambda(A)<\delta$ for $A\in\mathcal{B}(V)$ (resp. for $A\in\mathcal{B}(T)$). By hypothesis, f is $(x_n^*\circ\mathbf{m})$-measurable and hence by Theorem 21.6 (resp. by Theorem 21.5) f is Lusin $(x_n^*\circ\mathbf{m})$-measurable. Therefore, for each $n\in\mathbb{N}$, there exists $K_n\in\mathcal{C}$ such that $K_n\subset K$, $f|_{K_n}$ is continuous and $v(x_n^*\circ\mathbf{m})(K\backslash K_n)<\frac{\delta}{2^n c_n}$. Then $K_0=\bigcap_1^\infty K_n\in\mathcal{C}$, $f|_{K_0}$ is continuous and $\lambda(K\backslash K_0) \leq \sum_1^\infty c_n v(x_n^* \circ \mathbf{m})(K\backslash K_n) < \delta$. Hence $||\mathbf{m}||(K\backslash K_0) < \epsilon$. Therefore, f is Lusin $\mathbf{m}$-measurable. When $\mathbf{m}$ is defined on $\mathcal{B}(T)$, then by Theorem 21.5, f is $\mathbf{m}$-measurable. If $\mathbf{m}$ is defined on $\delta(\mathcal{C})$, then by hypothesis, f is $(x^* \circ \mathbf{m})$-measurable for $x^* \in H$ and hence, given $x^* \in H$, there exist N_{x^*}, M_{x^*} and B_{x^*} such that $B_{x^*} \in \mathcal{B}_c(T)$, $N_{x^*} \subset M_{x^*} \in \mathcal{B}_c(T)$ and $||\mathbf{m}||(M_{x^*}) = 0$ and such that $N(f) = B_{x^*} \cup N_{x^*}$. Hence $N(f)$ is σ-bounded and consequently, f is $\mathbf{m}$-measurable by Theorem 21.6.

In the sequel, $\mathcal{K}(T)$ is as in Notation 19.1 of [P11].

Theorem 22.3. Let $\mu_i : \delta(\mathcal{C}) \to \boldsymbol{K}$ be σ-additive and $\delta(\mathcal{C})$-regular for $i \in I$. Suppose $\sum_{i\in I} |\int_T \varphi d\mu_i|^p < \infty$ for each $\varphi \in \mathcal{K}(T)$ and for $1 \leq p < \infty$. Let $u : \mathcal{K}(T) \to l_p(I)$ be defined by $u(\varphi) = (\int_T \varphi d\mu_i)_{i\in I}$. Then u is a prolongable Radon operator on $\mathcal{K}(T)$. Let $\mathbf{m}_u$ be the representing measure of u. (See Definitions 19.5 and 19.6 and Theorem 19.9 of [P11].) Let $f : T \to \boldsymbol{K}$ belong to $\mathcal{L}_1(\mu_i)$ for $i \in I$. Then f is $\mathbf{m}_u$-integrable in T if and only if

$$\sum_{i\in I} |\int_U f d\mu_i|^p < \infty \qquad (22.3.1)$$

for each open Baire set U in T. In that case, $\int_T f d\mathbf{m}_u = (\int_T f d\mu_i)_{i\in I}$.

Let $p = 1$ and let $f \in \mathcal{L}_1(\mathbf{m}_u)$. If $\theta(\varphi) = \sum_{i\in I} \int_T \varphi d\mu_i$ for $\varphi \in \mathcal{K}(T)$, then $\theta \in \mathcal{K}(T)^*$, f is μ_θ-integrable and

$$\int_A f d\mu_\theta = \sum_{i\in I} \int_A f d\mu_i$$

for $A \in \mathcal{B}_c(T)$, where μ_θ is the complex Radon measure induced by θ in the sense of Definition 4.3 of [P1].

Proof. Let us recall from Notation 19.1 of [P11] that the topology of $\mathcal{K}(T)$ is the inductive limit locally convex topology on $C_c(T)$ induced by the family $(C_c(T, C), I_C)$ where $C_c(T, C)$ are provided with the topology τ_u of uniform convergence. Clearly, $C_c(T, C)$ are Banach spaces. Let $u : \mathcal{K}(T) \to \ell_p(I)$ be given by $u(\varphi) = (\int_T \varphi d\mu_i)_{i\in I}$. Clearly, u is linear. We claim that u has a closed graph. In fact, let $\varphi_\alpha \to \varphi$ in $\mathcal{K}(T)$. As $\mu_i \in \mathcal{K}(T)^*$ (see Section 5 of [P2]), $\int_T \varphi_\alpha d\mu_i \to \int_T \phi d\mu_i$ for each $i \in I$. Suppose $u(\varphi_\alpha) \to (f_i)_{i\in I} \in \ell_p(I)$. Then given $\epsilon > 0$, there exist $J \subset I$, J finite, and an α_0, such that $\sum_{i\in J} |\int_T \varphi_\alpha d\mu_i - f_i|^p < (\frac{\epsilon}{2})^p$ for $\alpha \geq \alpha_0$. Moreover, there exists $\alpha_1 \geq \alpha_0$ such that $\sum_{i\in J} |\int_T \varphi_\alpha d\mu_i - \int_T \varphi d\mu_i|^p < (\frac{\epsilon}{2})^p$ as $\int_T \varphi_\alpha d\mu_i \to \int_T \varphi d\mu_i$ for each i. Then

$$\begin{aligned}(\sum_{i\in J} |\int_T \varphi d\mu_i - f_i|^p)^{\frac{1}{p}} &\leq (\sum_{i\in J} |\int_T \varphi d\mu_i - \int_T \varphi_\alpha d\mu_i|^p)^{\frac{1}{p}} + (\sum_{i\in J} \int_T |\varphi_\alpha d\mu_i - f_i|^p)^{\frac{1}{p}} \\ &< \epsilon\end{aligned}$$

for $\alpha \geq \alpha_1$. Thus, for $i \in J$,

$$|\int_T \varphi d\mu_i - f_i| \leq (\sum_{j\in J} |\int_T \varphi d\mu_j - f_j|^p)^{\frac{1}{p}} < \epsilon.$$

Since ϵ is arbitrary, $\int_T \varphi d\mu_i = f_i$. If $i \notin J$, by the same argument with $J \cup \{i\}$ in place of J, we have $\int \varphi d\mu_i = f_i$ for each $i \in I$. Thus $(f_i)_i = u(\varphi)$ and hence the graph of u is closed.

Since $C_c(T, C)$ is a Banach space, any linear mapping from $C_c(T, C)$ into $\ell_p(I)$ with closed graph is continuous by the closed graph theorem (see Theorem 2.15 of [Ru2]) and hence by Problem C(i), Sec. 16, Ch. 5 of [KN], u is a continuous linear mapping.

Let $V \in \mathcal{V}$ and $u_V(\varphi) = u(\varphi)$ for $\varphi \in C_c(V)$.Then clearly $u_V : C_c(V) \to \ell_p(I)$ is continuous and its continuous extension $\widetilde{u_V} : C_0(V) \to \ell_p(I)$ is weakly compact by by Theorem 13 of [P5] or by Corollary 2 of [P6] since $c_0 \not\subset \ell_p(I)$ for $1 \leq p < \infty$ (as $\ell_1(I)$ is weakly sequentially complete and as $\ell_p(I)$ is reflexive for $1 \leq p < \infty$.) Hence u is a prolongable Radon operator on $\mathcal{K}(T)$ and hence by Theorem 19.9 of [P11] its representing measure $\mathbf{m}_u : \delta(\mathcal{C}) \to \ell_p(I)$, $1 \leq p < \infty$, is σ-additive and $\delta(\mathcal{C})$-regular and

$$u(\varphi) = \int_T \varphi d\mathbf{m}_u, \ \varphi \in C_c(T) \qquad (22.3.2)$$

where the integral is a (BDS)-integral.

For $1 < p < \infty$, let $H_I^{(p)} = \{(\alpha_i)_{i\in I} \in \ell_q(I) : \sum_{i\in I} |\alpha_i|^q \leq 1, \alpha_i = 0 \,\text{for}\, i \in I\backslash J, \,\text{where}\, J \subset I, J \,\text{finite}\}$ where $\frac{1}{p} + \frac{1}{q} = 1$. For $p = 1$, let $H_I^{(1)} = \{(\alpha_i)_{i\in I} \in \ell_\infty(I) : \sup_{i\in I} |\alpha_i| \leq 1, \alpha_i = 0 \,\text{for}\, i \in I\backslash J, \,\text{where}\, J \subset I, J \,\text{finite}\}$. Clearly, $H_I^{(p)}$ is a norm determining set for $\ell_p(I)$, $1 \leq p < \infty$.

Claim 1. Let $x^* = (\alpha_i)_{i\in I} \in H_I^{(p)}$, $1 \leq p < \infty$, where $\alpha_i = 0$ for $i \in I\backslash J_{x^*}$, $J_{x^*} \subset I$ and finite. Then $x^* \circ \mathbf{m}_u = \sum_{i\in I} \alpha_i\mu_i = \sum_{i\in J_{x^*}} \alpha_i\mu_i$.

In fact, by Theorem 11.8(v) and Remark 12.5 of [P10] and by (22.3.2) we have

$$\int_T \varphi d(x^* \circ \mathbf{m}_u) = x^* u(\varphi) = \sum_{i\in I} \alpha_i \int_T \varphi d\mu_i = \int_T \varphi d(\sum_{i\in J_{x^*}} \alpha_i\mu_i) \quad (22.3.3)$$

for $\varphi \in \mathcal{K}(T)$. Let $V \in \mathcal{V}$. Then, for $\varphi \in C_c(V)$, by (22.3.3) we have $\int_T \varphi d(x^* \circ (\mathbf{m}_u)_V = x^* u_V(\varphi) = x^* u(\varphi) = \int_T \varphi d(\sum_{i\in J_{x^*}} \alpha_i\mu_i)$, where $(\mathbf{m}_u)_V = \mathbf{m}_u|_{\mathcal{B}(V)}$. As $x^* \circ (\mathbf{m}_u)_V$ and $\mu_i|_{\mathcal{B}(V)}$ are σ-additive and $\mathcal{B}(V)$-regular, by the uniqueness part of the Riesz representation theorem we conclude that $x^* \circ (\mathbf{m}_u)_V = \sum_{i\in I} \alpha_i\mu_i|_{\mathcal{B}(V)}$. Since V is arbitrary in $\mathcal{V}$ and since $\delta(\mathcal{C}) = \bigcup_{V\in\mathcal{V}} \mathcal{B}(V)$, it follows that $x^* \circ \mathbf{m}_u = \sum_{i\in J_{x^*}} \alpha_i\mu_i = \sum_{i\in I} \alpha_i\mu_i$. Hence the claim holds.

Let $\varphi \in C_0(T)$. By hypothesis, $f \in \mathcal{L}_1(\mu_i)$ for $i \in I$ and φ is $\mathcal{B}_c(T)$-measurable by Theorem 51.B of [H] and is bounded. Hence $f\varphi \in \mathcal{L}_1(\mu_i)$ for $i \in I$. Let

$$\theta_i(\varphi) = \int_T f\varphi d\mu_i, \ \ \varphi \in C_0(T)$$

for $i \in I$. Then θ_i is a bounded linear functional on $C_0(T)$ and hence the complex Radon measure μ_{θ_i} induced by θ_i is a σ-additive $\mathcal{B}(T)$-regular scalar measure on $\mathcal{B}(T)$ by Theorems 3.3 and 4.6 of [P2]. On the other hand,

Claim (*). $\eta_i(\cdot) = \int_{(\cdot)} f d\mu_i$ is σ-additive on $\mathcal{B}_c(T)$ and η_i is $v(\mu_i)$-continuous, $i \in I$.(In symbols, $\eta_i \ll v(\mu_i)$.)

In fact, $v(\mu_i) : \mathcal{B}_c(T) \to [0, \infty]$ is σ-additive by Property 9, § 3, Ch. I of [Din] and $v(\mu_i)(E) = 0$, $E \in \mathcal{B}_c(T)$ implies $v(\eta_i)(E) = 0$. Then by Theorem 6.11 of [Ru1] (whose proof is valid for σ-rings too) we conclude that $v(\eta_i)$ is $v(\mu_i)$-continuous on $\mathcal{B}_c(T)$ and hence η_i is $v(\mu_i)$-continuous.

Therefore, $v(\eta_i) \ll v(\mu_i)$ on $\delta(\mathcal{C})$ and consequently, η_i, $i \in I$, are $\delta(\mathcal{C})$-regular. Moreover, for $\varphi \in C_0(T)$, we have

$$\int_T \varphi d\eta_i = \int_T \varphi f d\mu_i = \theta_i(\varphi) = \int_T \varphi d\mu_{\theta_i},\ i \in I. \qquad (22.3.4)$$

Thus, for $V \in \mathcal{V}$ and $\varphi \in C_c(V)$, we have $\int_T \varphi d\eta_i|_{\mathcal{B}(V)} = \int_T \varphi d\mu_{\theta_i}|_{\mathcal{B}(V)}$ and hence by the uniqueness part of the Riesz representation theorem, we have $\eta_i|_{\mathcal{B}(V)} = \mu_{\theta_i}|_{\mathcal{B}(V)}$. As V is arbitrary in $\mathcal{V}$, we conclude that

$$\eta_i|_{\delta(\mathcal{C})} = \mu_{\theta_i}|_{\delta(\mathcal{C})}. \qquad (22.3.5)$$

Since $v(\eta_i)(T) = \int_T |f| dv(\mu_i) < \infty$ by Proposition 2.11 of [P8] and by the hypothesis that $f \in \mathcal{L}_1(\mu_i)$ and since $\sigma(\delta(\mathcal{C})) = \mathcal{B}_c(T)$, we conclude that

$$\eta_i = \mu_{\theta_i}|_{\mathcal{B}_c(T)} \qquad (22.3.6)$$

for $i \in I$. Then by Theorem 2.4 of [P2], η_i is $\mathcal{B}_c(T)$-regular for $i \in I$.

Let $x^* = (\alpha_i) \in H_I^{(p)}$. Then there exists a finite set $J_{x^*} \subset I$ such that $\alpha_i = 0$ for $i \in I \backslash J_{x^*}$. Let $\Psi_{x^*} = \sum_{i \in J_{x^*}} \alpha_i \theta_i$. Then Ψ_{x^*} is a bounded linear functional on $C_0(T)$ and $\Psi_{x^*}(\varphi) = \sum_{i \in J_{x^*}} \alpha_i \theta_i(\varphi) = \int_T \varphi d(\sum_{i \in J_{x^*}} \alpha_i \mu_{\theta_i})$ by (22.3.4). Then arguing as in the proof of (22.3.5) and using (22.3.6), we have

$$\mu_{\Psi_{x^*}} = \sum_{i \in J_{x^*}} \alpha_i \mu_{\theta_i} \text{ on } \mathcal{B}(T) \text{ and } \mu_{\psi_{x^*}}|_{\mathcal{B}_c(T)} = \sum_{i \in J_{x^*}} \alpha_i \eta_i. \qquad (22.3.7)$$

Claim 2. $\sup_{x^* \in H_I^{(p)}} v(\mu_{\Psi_{x^*}}, \mathcal{B}(T))(T) = M \text{ (say)} < \infty$ for $1 \le p < \infty$.

In fact, let U be an open Baire set in T and let $1 < p < \infty$. Then by hypothesis (22.3.1), by (22.3.7) and by Hölder's inequality, we have

$$\begin{aligned}
\sup_{x^*\in H_I^{(p)}} |\mu_{\Psi_{x^*}}(U)| &= \sup_{x^*=(\alpha_i)_i\in H_I^{(p)}} |\sum_{i\in J_{x^*}} \alpha_i\eta_i(U)| \\
&= \sup_{x^*\in H_I^{(p)}} |x^*(\int_U f d\mu_i)_{i\in I}| \\
&\leq \sup_{x^*\in H_I^{(p)}} |x^*|_q(\sum_{i\in I} |\int_U f d\mu_i|^p)^{\frac{1}{p}} \\
&\leq (\sum_{i\in I} |\int_U f d\mu_i|^p)^{\frac{1}{p}} < \infty \qquad (22.3.8)
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. If $p = 1$, by (22.3.1) and by (22.3.7) we have

$$\begin{aligned}
\sup_{x^*\in H_I^{(1)}} |\mu_{\Psi_{x^*}}(U)| &= \sup_{x^*=(\alpha_i)_i\in H_I^{(1)}} |\sum_{i\in J_{x^*}} \alpha_i\eta_i(U)| \\
&\leq \sup_{x^*\in H_I^{(1)}} |x^*(\int_U f d\mu_i)_{i\in I}| \\
&= \sup_{x^*\in H_I^{(1)}, x^*=(\alpha_i)_{i\in I}} \sum_{i\in I} |\alpha_i \int_U f d\mu_i| \\
&\leq \sum_{i\in I} |\int_U f d\mu_i| < \infty. \qquad (22.3.8')
\end{aligned}$$

As $\{\mu_{\Psi_{x^*}} : x^* \in H_I^{(p)}\} \subset M(T)$ for $1 \leq p < \infty$, the claim holds by (22.3.8) and (22.3.8') and by Corollary 18.5 of [P11].

Claim 3. Given $\varphi \in C_0(T)$ and $\epsilon > 0$, there exists a simple function s as a complex linear combination of the characteristic functions of relatively compact open Baire sets in T such that

$$||s - \varphi||_T < \frac{\epsilon}{2M} \qquad (22.3.9)$$

where M is as in Claim 2.

In fact, in the proof of Lemma 18.20(i) of [P11], each of the sets $E_{i,n}$ is a difference of two open sets $U_{i,n}$, $V_{i,n}$ in T which are F_σ and in fact, are σ-compact and relatively compact as $\operatorname{supp}\varphi$ is compact. Then by Lemma 18.3 of [P11], $U_{i,n}$ and $V_{i,n}$ are relatively compact open Baire sets in T. Then, the functions (s'_n) in the proof of Lemma 18.20(i) of [P11] are complex linear combinations of the characteristic functions of relatively compact open Baire sets in T. As $s'_n \to \varphi$ uniformly in T, the claim holds.

Claim 4. Let $\varphi \in C_0(T)$. Then

$$\sum_{i\in I} |\int_T f\varphi d\mu_i|^p < \infty. \qquad (22.3.10)$$

In fact, given $\epsilon > 0$, choose s as in Claim 3. By hypothesis (22.3.1), $\sum_{i\in I} |\int_U sfd\mu_i|^p < \infty$. Then there exists a finite set $J_0 \subset I$ such that

$$\sum_{I\backslash J_0} |\int_T sfd\mu_i|^p < (\frac{\epsilon}{2})^p. \qquad (22.3.11)$$

Let $I_1 = I\backslash J_0$. Let $H^{(p)}_{I_1} = \{(\alpha_i)_{i\in I_1} :$ there exists a finite set $J \subset I_1$ such that $\alpha_i = 0$ for $i \in I_1\backslash J$ and $||(\alpha_i)_{i\in I_1}||_q \leq 1\}$ where $\frac{1}{p} + \frac{1}{q} = 1$ when $1 < p < \infty$; and $q = \infty$ when $p = 1$. Let $x^* = (\alpha_i)_{i\in I_1} \in H^{(p)}_{I_1}$ be fixed. Then there exists a finite set $J_{x^*} \subset I_1$ such that $\alpha_i = 0$ for $i \in I_1\backslash J_{x^*}$. Let

$$\Phi_{x^*}(\varphi) = \sum_{i\in J_{x^*}} \alpha_i\theta_i(\varphi).$$

Then by (22.3.4) we have

$$\Phi_{x^*}(\varphi) = \sum_{i\in J_{x^*}} \alpha_i\mu_{\theta_i}(\varphi) = \sum_{i\in J_{x^*}} \int_T \alpha_i f\varphi d\mu_i \qquad (22.3.12)$$

for $\varphi \in C_0(T)$. Then Φ_{x^*} is a bounded linear functional on $C_0(T)$ and

$$|\Phi_{x^*}(\varphi)| \leq |\int_T(\varphi - s)d\mu_{\Phi_{x^*}}| + |\int_T sd\mu_{\Phi_{x^*}}| \leq ||\varphi - s||_T v(\mu_{\Phi_{x^*}}, \mathcal{B}(T))(T) + |\sum_{i\in J_{x^*}} \alpha_i \int_T sfd\mu_i|$$

since $\mu_{\Phi_{x^*}} = \sum_{i\in J_{x^*}} \alpha_i\mu_{\theta_i} = \sum_{i\in J_{x^*}} \alpha_i\eta_i$ on $\mathcal{B}_c(T)$ by (22.3.6) and since s is a $\mathcal{B}_c(T)$-simple function. Taking $y^* = (\alpha_i)_{i\in I}$ with $\alpha_i = 0$ for $i \in I\backslash J_{x^*}$, we observe that Φ_{x^*} is the same as Ψ_{y^*} defined before Claim 2 and hence by Claim 2 we have $\sup_{x^*\in H^{(p)}_{I_1}} v(\mu_{\Phi_{x^*}}, \mathcal{B}(T))(T) \leq M$ for $1 \leq p < \infty$ where M is as in Claim 2. Hence by (22.3.9) and (22.3.11) we have

$$\begin{aligned}
|\Phi_{x^*}(\varphi)| &\leq ||\varphi - s||_T \cdot v(\mu_{\Phi_{x^*}}, \mathcal{B}(T))(T) + |(\sum_{i\in J_{x^*}} \alpha_i \int_T sfd\mu_i| \\
&\leq \frac{\epsilon}{2M} \cdot M + |x^*(\int_T sfd\mu_i)_{i\in I_1}| \\
&\leq \frac{\epsilon}{2} + ||x^*||_q||(\int_T sfd\mu_i)_{i\in I_1}||_p \\
&< \epsilon.
\end{aligned}$$

Varying $x^* \in H^{(p)}_{I_1}$, we have $\sup_{x^*\in H^{(p)}_{1}} |\Phi_{x^*}(\varphi)| \leq \epsilon$. As $H^{(p)}_{I_1}$ is a norm determining set for $\ell_p(I_1)$ for $1 \leq p < \infty$ and as $\Phi_{x^*}(\varphi) = x^*(\int_T f\varphi d\mu_i)_{i\in I_1}$ by (22.3.12), we have

$$(\sum_{i\in I_1} |\int_T f\varphi d\mu_i|^p)^{\frac{1}{p}} = ||(\int_T f\varphi d\mu_i)_{i\in I_1}||_p = \sup_{x^*\in H^{(p)}_{I_1}} |\Phi_{x^*}(\varphi)| \leq \epsilon$$

for $1 \leq p < \infty$. Hence the claim holds.

Then by Claim 4, the mapping $\xi : C_0(T) \to \ell_p(I)$ given by

$$\xi(\varphi) = (\int_T f\varphi d\mu_i)_{i\in I} = (\int_T \varphi d\eta_i)_{i\in I}$$

is well defined and linear. Moreover, by the closed graph theorem ξ is continuous. Since $c_0 \not\subset \ell_p(I)$ for $1 \leq p < \infty$, ξ is weakly compact by Theorem 13 of [P5] for $1 \leq p < \infty$. Then by Theorems 2 and 6 of [P5] its representing measure $\mathbf{m}_\xi : \mathcal{B}(T) \to \ell_p(I)^{**}$ has range in $\ell_p(I)$, and is σ-additive and $\mathcal{B}(T)$-regular.

Claim 5. Let $x^* = (\alpha_i)_{i\in I} \in H_I^{(p)}$, $1 \leq p < \infty$, so that there exists a finite set $J_{x^*} \subset I$ such that $\alpha_i = 0$ for $i \in I \backslash J_{x^*}$. Then

$$(x^* \circ \mathbf{m}_\xi)|_{\mathcal{B}_c(T)} = \sum_{i\in J_{x^*}} \alpha_i\eta_i = \sum_{i\in I} \alpha_i\eta_i.$$

In fact, for $\varphi \in C_0(T)$, by Theorem 1 of [P5] we have

$$\int_T \varphi d(x^* \circ \mathbf{m}_\xi) = x^*\xi(\varphi) = \sum_{i\in J_{x^*}} \alpha_i \int_T \varphi d\eta_i = \int_T \varphi d(\sum_{i\in J_{x^*}} \alpha_i\eta_i);$$

$(x^* \circ \mathbf{m}_\xi)|_{\mathcal{B}_c(T)}$ is $\mathcal{B}_c(T)$-regular since $(\mathbf{m}_\xi)|_{\mathcal{B}_c(T)}$ is $\mathcal{B}_c(T)$-regular by Theorem 7(xxiii) of [P5] and $\sum_{i\in J_{x^*}} \alpha_i\eta_i$ is $\mathcal{B}_c(T)$-regular as observed after (22.3.6). Consequently, by the uniqueness part of the σ-Borel version of the Riesz representation theorem the claim holds.

By hypothesis, f is μ_i-measurable for $i \in I$ and hence f is $\sum_{i\in J_{x^*}} \alpha_i\mu_i$-measurable for $x^* = (\alpha_i)_{i\in I} \in H_I^{(p)}$ for $1 \leq p < \infty$, where $\alpha_i = 0$ for $i \in I\backslash J_{x^*}$, $J_{x^*} \subset I$ and J_{x^*} is finite. Since $\eta_i \ll v(\mu_i)$, $i \in I$ by Claim (*), f is also η_i-measurable for $i \in I$. Hence by Claim 5, f is $(x^* \circ \mathbf{m}_\xi)|_{\mathcal{B}_c(T)}$-measurable for $x^* \in H_I^{(p)}$. Since $\mathbf{m}_\xi$ is $\ell_p(I)$-valued σ-additive and Borel regular for $1 \leq p < \infty$, by Theorem 20.12, $\mathbf{m}_\xi|_{\delta(\mathcal{C})}$ is $\ell_p(I)$-valued σ-additive and $\delta(\mathcal{C})$-regular for $1 \leq p < \infty$. As $H_I^{(p)}$ is a norm determining set for $\ell_p(I)$, then by the last part of Lemma 22.2, f is $\mathbf{m}_\xi|_{\delta(\mathcal{C})}$-measurable as well as Lusin $\mathbf{m}_\xi|_{\delta(\mathcal{C})}$-measurable.

Let $\epsilon > 0$. As $\mathbf{m}_\xi$ is $\mathcal{B}(T)$-regular, there exists $K \in \mathcal{C}$ such that $||\mathbf{m}_\xi||(T\backslash K) < \frac{\epsilon}{2}$. As f is Lusin $\mathbf{m}_\xi|_{\delta(\mathcal{C})}$-measurable, there exists $K_0 \in \mathcal{C}$ with $K_0 \subset K$ such that $f|_{K_0}$ is continuous and $||\mathbf{m}_\xi||(K\backslash K_0) < \frac{\epsilon}{2}$. Then $||\mathbf{m}_\xi||(T\backslash K_0) < \epsilon$ and $f\chi_{K_0}$ is bounded and $\mathcal{B}(K_0)$-measurable, as it is continuous on the compact K_0. Consequently, $f\chi_{K_0}$ is a bounded $\mathcal{B}_c(T)$-measurable function with compact support. As u is prolongable with the reprsenting measure $\mathbf{m}_u$, by (20) of Theorem 19.12 of [P11] and by Theorem 3.5(v) and Remark 4.3 of [P8], $f\chi_{K_0} \in \mathcal{L}_1(\mathbf{m}_u)$. By Claim 1 and by the hypothesis that $f \in \mathcal{L}_1(\mu_i)$ for $i \in I$, f is $(x^* \circ \mathbf{m}_u)$-measurable for each $x^* \in H_I^{(p)}$, $1 \leq p < \infty$ and as observed in the beginning of the proof, $\mathbf{m}_u$ is $\ell_p(I)$-valued, σ-additive and $\delta(\mathcal{C})$-regular. As $H_I^{(p)}$ is a norm determining set for $\ell_p(I)$, $1 \leq p < \infty$, by Lemma 22.2, f is $\mathbf{m}_u$-measurable. Consequently, $f - f\chi_{K_0}$ is also $\mathbf{m}_u$-measurable. Now by Claims 1 and 5, by Lemma 22.1(i) and by the fact that $\int_{N(f)\backslash K_0} |f| dv(\sum_{i\in J_{x^*}} \alpha_i\mu_i) = v(\int_{N(f)\backslash K_0)} fd(\sum_{i\in J_{x^*}} \alpha_i\mu_i)) = v(\sum_{i\in J_{x^*}} \alpha_i\eta_i)(N(f)\backslash K_0)$ (by Proposition 2.11 of [P8]), we have

$$\begin{aligned}
\sup_{x^*\in H_I^{(p)}} \int_T |f - f\chi_{K_0}| dv(x^*\circ \mathbf{m}_u) &= \sup_{x^*\in H_I^{(p)}} \int_{N(f)\backslash K_0} |f| dv(x^*\circ \mathbf{m}_u) \\
&= \sup_{x^*=(\alpha_i)_i\in H_I^{(p)}} \int_{N(f)\backslash K_0} |f| dv(\sum_{i\in J_{x^*}} \alpha_i\mu_i) \\
&= \sup_{x^*=(\alpha_i)_{i\in I}\in H_I^{(p)}} v(\sum_{i\in J_{x^*}} \alpha_i\eta_i)(N(f)\backslash K_0) \\
&= \sup_{x^*\in H_I^{(p)}} v(x^*\circ \mathbf{m}_\xi)(N(f)\backslash K_0) \\
&\le \sup_{x^*\in H_I^{(p)}} v(x^*\circ \mathbf{m}_\xi)(T\backslash K_0) \\
&\le ||\mathbf{m}_\xi||(T\backslash K_0) < \epsilon
\end{aligned}$$

for $1 \le p < \infty$. Since f is $\mathbf{m}_u$-measurable, $f\chi_{K_0} \in \mathcal{L}_1(\mathbf{m}_u)$, $H_I^{(p)}$ is a norm determining set for $\ell_p(I)$ for $1 \le p < \infty$ and $\epsilon > 0$ is arbitrary, by Lemma 22.1(ii) we conclude that $f \in \mathcal{L}_1(\mathbf{m}_u)$. Hence the condition (22.3.1) is sufficient.

Let $f \in \mathcal{L}_1(\mathbf{m}_u)$. Let $x_i^* = (\alpha_j)_{j\in I} \in H_I^{(p)}$, where $\alpha_i = 1$ and $\alpha_j = 0$, $j \ne i$. Then by Claim 1, $x_i^* \circ \mathbf{m}_u = \mu_i$ and hence by Theorem 19.11(iii) of [P11], $f \in \mathcal{L}_1(\mu_i)$ and $x_i^*(\int_T fd\mathbf{m}_u) = \int_T fd(x_i^*\circ \mathbf{m}_u) = \int_T fd\mu_i$. Hence

$$\int_T fd\mathbf{m}_u = (\int_T fd\mu_i)_{i\in I} \in \ell_p(I) \qquad (22.3.13)$$

for $1 \le p < \infty$.

Conversely, let $f \in \mathcal{L}_1(\mathbf{m}_u)$ and let U be an open Baire set in T. Then by Theorem 3.5(vi) and by Remark 4.3 of [P8], $f\chi_U \in \mathcal{L}_1(\mathbf{m}_u)$ and hence by (22.3.13), $\int_T f\chi_U d\mathbf{m}_u = (\int_U fd\mu_i)_{i\in I} \in \ell_p(I)$. Therefore $\sum_{i\in I} |\int_U fd\mu_i|^p < \infty$ for $1 \le p < \infty$. Thus the condition (22.3.1) is also necessary.

Let $p = 1$, $f \in \mathcal{L}_1(\mathbf{m}_u)$ and $x^* = (\alpha_i)_{i\in I} \in \ell_\infty(I)$, where $\alpha_i = 1$ for each i. Then θ given in the last part of the theorem is the same as x^*u and hence $\theta \in \mathcal{K}(T)^*$. Then by Theorem 19.11 of [P11], $\mu_\theta = \mathbf{m}_{x^*u}$ considering x^*u as a scalar valued prolongable operator. Moreover, by the same theorem, we have $f \in \mathcal{L}_1(x^*\mathbf{m}_u) = \mathcal{L}_1(\mathbf{m}_{x^*u}) = \mathcal{L}_1(\mu_\theta)$ and

$$\int_A fd\mu_\theta = \int_A fd\mathbf{m}_{x^*u} = x^*(\int_A fd\mathbf{m}_u) = x^*((\int_A fd\mu_i)_{i\in I}) = \sum_{i\in I}\int_A fd\mu_i$$

for each $A \in \mathcal{B}_c(T)$.

This completes the proof of the theorem.

The above theorem for the case $I = \mathbb{N}$ and $p = 1$ is used in the proof of the following result, the last part of which strengthens Theorem 4.2 of [P8] when $\mathcal{P} = \delta(\mathcal{C})$ and $\mathbf{m}$ is a Banach space valued σ-additive $\mathcal{P}$-regular measure on $\mathcal{P}$.

Theorem 22.4. Let X be a Banach space and let $\mathbf{m} : \delta(\mathcal{C}) \to X$ be σ-additive and $\delta(\mathcal{C})$-regular. Let H be a norm determining set for X with the Orlicz property. Then a function $f : T \to \mathbb{K}$ is $\mathbf{m}$-integrable in T if and only if $f \in \mathcal{L}_1(x^* \circ \mathbf{m})$ for each $x^* \in H$ and, for each open Baire set U in T, there exists a vector $x_U \in X$ such that

$$x^*(x_U) = \int_U f d(x^* \circ \mathbf{m}) \qquad (22.4.1)$$

for $x^* \in H$. In that case, $f\varphi \in \mathcal{L}_1(\mathbf{m})$ for each $\varphi \in C_0(T)$ and the mapping $\Psi : C_0(T) \to X$ given by $\Psi(\varphi) = \int_T f\varphi d\mathbf{m}$ is a weakly compact operator. Consequently, f is $\mathbf{m}$-integrable in T if and only if $f \in \mathcal{L}_1(x^* \circ \mathbf{m})$ for $x^* \in X^*$ and (22.4.1) holds for each $x^* \in X^*$ and for each open Baire set U in T.

Proof. If $f \in \mathcal{L}_1(\mathbf{m})$, then f is (KL) $\mathbf{m}$-integrable in T by Theorem 4.2 of [P8] and hence the conditions hold.

Conversely, let the conditions hold. Let $< H >$ be the vector space spanned by H and let F be the norm closure of $< H >$ in X^*. By hypothesis, for each open Baire set U in T there exists $x_U \in X$ such that (22.4.1) holds for $x^* \in H$ and consequently,

$$x^*(x_U) = \int_U f d(x^* \circ \mathbf{m}) \qquad (22.4.2)$$

for $x^* \in < H >$.

Claim 1. (22.4.2) holds for each $x^* \in F$ and for each open Baire set U in T.

In fact, given $x^* \in F$, there exists a sequence $(x_n^*)_1^\infty \subset < H >$ such that $x^* = \sum_1^\infty x_n^*$ with $\sum_1^\infty |x_n^*| < \infty$. Then, for $\varphi \in \mathcal{K}(T)$, we have

$$\sum_1^\infty |\int_T \varphi d(x_n^* \circ \mathbf{m})| = \sum_1^\infty |x_n^*(\int_T \varphi d\mathbf{m})| \leq |\int_T \varphi d\mathbf{m}|(\sum_1^\infty |x_n^*|) < \infty \quad (22.4.3)$$

since $\varphi \in \mathcal{L}_1(\mathbf{m})$ by Theorem 3.5(v) and Remark 4.3 of [P8] and since Theorem 3.5(viii) of [P8] applies by the same remark. Moreover, by (22.4.2) we have

$$\sum_1^\infty |\int_U f d(x_n^* \circ \mathbf{m})| = \sum_1^\infty |x_n^*(x_U)| \leq (\sum_1^\infty |x_n^*|)|x_U| < \infty. \qquad (22.4.4)$$

Clearly, $x^* \circ \mathbf{m} = \sum_1^\infty x_n^* \circ \mathbf{m}$ as $\mathbf{m}$ has range in X. As $\mathbf{m}$ is σ-additive and $\delta(\mathcal{C})$-regular on $\delta(\mathcal{C})$, $x^* \circ \mathbf{m}$ is σ-additive and $\delta(\mathcal{C})$-regular on $\delta(\mathcal{C})$.

The mapping $u : \mathcal{K}(T) \to \ell_1(\mathbb{N})$ given by

$$u(\varphi) = (\int_T \varphi d(x_n^* \circ \mathbf{m}))_{n=1}^\infty$$

is well defined by (22.4.3) and is clearly linear. By hypothesis $f \in \mathcal{L}_1(x_n^* \circ \mathbf{m})$ for $n \in \mathbb{N}$ and by (22.4.4), the complex measures $(x_n^* \circ \mathbf{m})_{n=1}^\infty$ satisfy the hypotheses of Theorem 22.3 for $p = 1$ and $I = \mathbb{N}$ and consequently, u is a prolongable Radon operator and $f \in \mathcal{L}_1(\mathbf{m}_u)$. By the last part of the said theorem, $\theta : \mathcal{K}(T) \to \mathbb{K}$, given by $\theta(\varphi) = \sum_{n=1}^\infty \int_T \varphi d(x_n^* \circ \mathbf{m})$, belongs to $\mathcal{K}(T)^*$, f is μ_θ-integrable and

$$\int_U f d\mu_\theta = \sum_{n=1}^\infty \int_U f d(x_n^* \circ \mathbf{m}) \qquad (22.4.5)$$

for each open Baire set U in T and for the set $U = N(f)$ since $f \in \mathcal{L}_1(\mathbf{m}_u)$ so that $N(f) \in \widetilde{\mathcal{B}_c(T)}$ with respect to $||\mathbf{m}||_u$.

Now $\int_T \varphi d\mu_\theta = \theta(\varphi) = \sum_{n=1}^\infty \int_T \varphi d(x_n^* \circ \mathbf{m}) = \int_T \varphi d(x^* \circ \mathbf{m})$ for $\phi \in \mathcal{K}(T)$, since $| \int_T \varphi d(x^* \circ \mathbf{m}) - \int_T \varphi d(\sum_1^k x_n^* \circ \mathbf{m})| = |x^* - \sum_1^k x_n^*|| \int_T \varphi d\mathbf{m}| \to 0$ as $k \to \infty$. Since $x^* \circ \mathbf{m}$ is σ-additive and $\delta(\mathcal{C})$-regular, and since $\mu_\theta|_{\delta(\mathcal{C})}$ is σ-additive and $\delta(\mathcal{C})$-regular by Theorem 4.4(i) of [P2], by an argument based on the uniqueness part of the Riesz representation theorem which is similar to that in the proof of Claim 1 in the proof of Theorem 22.3 we have $(x^* \circ \mathbf{m}) = \mu_\theta|_{\delta(\mathcal{C})}$. Then by (22.4.2) and (22.4.5) we have

$$\int_U f d(x^* \circ \mathbf{m}) = \int_U f d\mu_\theta = \sum_{n=1}^\infty \int_U f d(x_n^* \circ \mathbf{m}) = \sum_{n=1}^\infty x_n^*(x_U) = x^*(x_U)$$

for any open Baire set U in T and hence Claim 1 holds. Moreover, as f is μ_θ-integrable and as $\mu_\theta|_{\delta(\mathcal{C})} = x^* \circ \mathbf{m}$, f is $(x^* \circ \mathbf{m})$-integrable and hence $\int_T |f| dv(x^* \circ \mathbf{m}) < \infty$. Since x^* is arbitrary in F, it follows that $f \in \mathcal{L}_1(x^* \circ \mathbf{m})$ and hence

$$\int_T |f| dv(x^* \circ \mathbf{m}) < \infty \qquad (22.4.6)$$

for $x^* \in F$.

Let $\mathcal{F}$ be the vector space spanned by the characteristic functions of open Baire sets in T. Then for each $g \in \mathcal{F}$, by Claim 1 there exists $x_g \in X$ such that

$$x^*(x_g) = \int_T f g d(x^* \circ \mathbf{m}) \qquad (22.4.7)$$

for each $x^* \in F$. Let $\mathcal{G} = \{x_g : g \in \mathcal{F}, ||g||_T \leq 1\}$.

Claim 2. $\sup_{x_g \in \mathcal{G}} |x_g| = M$ (say) $< \infty$.

In fact, for $x^* \in F$, by (22.4.6) and (22.4.7) we have

$$\sup_{x_g \in \mathcal{G}} |x^*(x_g)| = \sup_{x_g \in \mathcal{G}} |\int_T fgd(x^* \circ \mathbf{m})| \leq \int_T |f| dv(x^* \circ \mathbf{m}) < \infty.$$

Hence $\mathcal{G}$ is $\sigma(X,F)$-bounded. Since H is a norm determining set for X, by Lemma 18.13 of [P11] we have $|x^*| \leq 1$ for $x^* \in H$ and hence $|x| = \sup_{|x^*| \leq 1} |x^*(x)| \geq \sup_{x^* \in F, |x^*| \leq 1} |x^*(x)| \geq \sup_{x^* \in H} |x^*(x)| = |x|$ for $x \in X$ (since $H \subset F$). Hence X can be considered as a subspace of F^* with the restriction of the norm of F^*. Then by the Banach-Steinhaus theorem applied to the Banach space F, the set $\mathcal{G}$ is norm bounded and hence the claim holds.

For $g \in \mathcal{F}$, let $\Phi(g) = x_g$. By (22.4.7) and by the hypothesis that H is norm determining, $\Phi : \mathcal{F} \to X$ is well defined and linear. By Claim 2, Φ is continuous. Hence Φ has a unique X-valued continuous linear extension $\hat{\Phi}$ on the closure $\bar{\mathcal{F}}$ of $\mathcal{F}$ in the Banach space of all bounded scalar functions on T, with the supremum norm. As $C_0(T) \subset \bar{\mathcal{F}}$ by Claim 3 in the proof of Theorem 22.3, define $\Phi_0 = \hat{\Phi}|_{C_0(T)}$. Then $\Phi_0 : C_o(T) \to X$ is linear and continuous and hence by Theorem 1 of [P5] its representing measure $\boldsymbol{\eta}$ is given by $\Phi_0^{**}|_{\mathcal{B}(T)}$. Moreover, by the same theorem, $x^* \circ \boldsymbol{\eta} \in M(T)$ for each $x^* \in X^*$ and

$$x^*\Phi_0(\varphi) = \int_T \varphi d(x^* \circ \boldsymbol{\eta}) \text{ for } \varphi \in C_0(T). \qquad (22.4.8)$$

By hypothesis, $f \in \mathcal{L}_1(x^* \circ \mathbf{m})$ for $x^* \in H$ and hence $\nu_{x^*}(\cdot) = \int_{(\cdot)} fd(x^* \circ \mathbf{m})$ is σ-additive on $\mathcal{B}_c(T)$ for $x^* \in H$. Clearly, $\nu_{x^*} \ll v(x^* \circ \mathbf{m})$. Then f is ν_{x^*}-measurable since f is $(x^* \circ \mathbf{m})$-measurable and ν_{x^*} is $\delta(\mathcal{C})$-regular for $x^* \in H$.

Let $\varphi \in C_0(T)$. Then there exists $(g_n)_1^\infty \subset \mathcal{F}$ such that $g_n \to \varphi$ uniformly in T so that $\Phi_0(\varphi) = \lim_n \Phi(g_n)$. Then by (22.4.7) and by the definition of Φ we have

$$x^*\Phi_0(\varphi) = \lim_n x^*\Phi(g_n) = \lim_n x^*(x_{g_n}) = \lim_n \int_T fg_n d(x^* \circ \mathbf{m}) \qquad (22.4.9)$$

for $x^* \in F$. On the other hand, as φ is $\mathcal{B}_c(T)$-measurable and bounded, $f\varphi \in \mathcal{L}_1(x^* \circ \mathbf{m})$ for $x^* \in H$ and by (22.4.6)

$$|\int_T f\varphi d(x^* \circ \mathbf{m}) - \int_T fg_n d(x^* \circ \mathbf{m})| \leq ||\varphi - g_n||_T (\int_T |f| dv(x^* \circ \mathbf{m}) \to 0$$

as $n \to \infty$ for each $x^* \in H$. Hence by (22.4.9) and (22.4.8) we have

$$\begin{aligned} \int_T \varphi d\nu_{x^*} = \int_T f\varphi d(x^* \circ \mathbf{m}) &= \lim_n \int_T fg_n d(x^* \circ \mathbf{m}) \\ &= x^*\Phi_0(\varphi) = \int_T \varphi d(x^* \circ \boldsymbol{\eta}) \qquad (22.4.10) \end{aligned}$$

for $\varphi \in C_0(T)$ and $x^* \in H$. As $x^* \circ \boldsymbol{\eta} \in M(T)$ and as ν_{x^*} is σ-additive and $\delta(\mathcal{C})$-regular for $x^* \in H$, by an argument similar to that in the proof of Claim 1 in the proof of Theorem 22.3 and by (22.4.10), we have

$$(x^* \circ \boldsymbol{\eta})|_{\delta(\mathcal{C})} = \nu_{x^*} \qquad (22.4.11)$$

for $x^* \in H$. As $v(x^* \circ \boldsymbol{\eta}, \mathcal{B}(T))(T) < \infty$, it follows that

$$v(\nu_{x^*}, \mathcal{B}_c(T))(\cdot) = v(x^* \circ \boldsymbol{\eta}, \mathcal{B}(T))|_{\mathcal{B}_c(T)}(\cdot) \qquad (22.4.12)$$

for $x^* \in H$. Consequently, as f is ν_{x^*}-measurable, it is $(x^* \circ \boldsymbol{\eta})|_{\mathcal{B}_c(T)}$-measurable for $x^* \in H$ and consequently, f is $(x^* \circ \boldsymbol{\eta})$-measurable.

Let U be an open Baire set in T. Then by §14 of [Din] there exists an increasing sequence (φ_n) of functions in $C_c(T)$ such that $\varphi_n \nearrow \chi_U$. Then by LDCT, by (22.4.1), (22.4.10) and (22.4.12) we have

$$\begin{aligned}(x^* \circ \boldsymbol{\eta})(U) &= \int_T \chi_U d(x^* \circ \boldsymbol{\eta}) = \lim_n \int_T \varphi_n d(x^* \circ \boldsymbol{\eta}) \\ &= \lim_n \int_T \varphi_n d\nu_{x^*} = \lim_n \int_T \varphi_n f d(x^* \circ \mathbf{m}) \\ &= \int_T \chi_U f d(x^* \circ \mathbf{m}) = x^*(x_U) \qquad (22.4.13)\end{aligned}$$

for $x^* \in H$. Then by Theorem 18.14 of [P11], Φ_0 is weakly compact and hence by Theorem 2 of [P5], $\boldsymbol{\eta}$ is σ-additive and has range in X and by Theorem 6(xix) of [P5], $\boldsymbol{\eta}$ is $\mathcal{B}(T)$-regular. (Note that only here we use the hypothesis that H has the Orlicz proprty to assert that Φ_0 is weakly compact.) Then, as f is $(x^* \circ \boldsymbol{\eta})$-measurable for $x^* \in H$, by the last part of Lemma 22.2 f is $\boldsymbol{\eta}$-measurable as well as Lusin $\boldsymbol{\eta}$-measurable. Given $\epsilon > 0$, by the $\mathcal{B}(T)$-regularity of $\boldsymbol{\eta}$ there exists $K \in \mathcal{C}$ such that $||\boldsymbol{\eta}||(T\backslash K) < \frac{\epsilon}{2}$. As f is Lusin $\boldsymbol{\eta}$-measurable, there exists $K_0 \in \mathcal{C}$ such that $K_0 \subset K$, $f|_{K_0}$ is continuous, and $||\boldsymbol{\eta}||(K\backslash K_0) < \frac{\epsilon}{2}$. Then

$$||\boldsymbol{\eta}||(T\backslash K_0) < \epsilon. \qquad (22.4.14)$$

Moreover, $f\chi_{K_0}$ is bounded and $\mathcal{B}_c(T)$-measurable with compact support so that $f\chi_{K_0} \in \mathcal{L}_1(\mathbf{m})$ by Theorem 3.5 and Remark 4.3 of [P8]. As f is $(x^* \circ \mathbf{m})$-measurable for $x^* \in H$, $f - f\chi_{K_0}$ is also $(x^* \circ \mathbf{m})$-measurable for $x^* \in H$. Moreover, as H is a norm determining set, by the last part of Lemma 22.2 $f - f\chi_{K_0}$ is $\mathbf{m}$-measurable. Then by (22.4.12), (22.4.14), by the fact that $v(\nu_{x^*})(\cdot) = \int_{(\cdot)} |f| dv(x^* \circ \mathbf{m})$ and by Lemma 22.1(i) we have

$$\begin{aligned}\sup_{x^* \in H} \int_T |f - f\chi_{K_0}| dv(x^* \circ \mathbf{m}) &= \sup_{x^* \in H} \int_{N(f)\backslash K_0} |f| dv(x^* \circ \mathbf{m}) \\ &= \sup_{x^* \in H} v(\nu_{x^*})(N(f)\backslash K_0) \\ &= \sup_{x^* \in H} v(x^* \circ \boldsymbol{\eta})(N(f)\backslash K_0) \\ &= ||\boldsymbol{\eta}||(N(f)\backslash K_0) \le ||\boldsymbol{\eta}||(T\backslash K_0) < \epsilon.\end{aligned}$$

Consequently, as $f\chi_{K_0} \in \mathcal{L}_1(\mathbf{m})$, by Lemma 22.1(ii), $f \in \mathcal{L}_1(\mathbf{m})$. Hence the conditions are also sufficient.

If $f \in \mathcal{L}_1(\mathbf{m})$, then by Theorem 3.5(vii) and Remark 4.3 of [P8], $f\varphi \in \mathcal{L}_1(\mathbf{m})$ for each $\varphi \in C_0(T)$ and hence $\Psi : C_0(T) \to X$ given by $\Psi(\varphi) = \int_T \varphi f d\mathbf{m}$ is well defined for $\varphi \in C_0(T)$. Consequently, by (22.4.10) and by (viii) of the said theorem in [P8] we have

$$x^*\Psi(\varphi) = \int_T f\varphi d(x^* \circ \mathbf{m}) = x^*\Phi_0(\varphi)$$

for $x^* \in H$ and for $\varphi \in C_0(T)$. As H is norm determining, it follows that $\Psi = \Phi_0$ and hence $\Psi$is weakly compact.

If $H = \{x^* \in X^* : |x^*| \le 1\}$, then H is a norm determining set for X and has the Orlicz property by the Orlicz-Pettis theorem. Hence the last part holds by the first part.

This completes the proof of the theorem.

The following result which is deduced from the last part of Theorem 22.4 improves Theorem 12.2(i) of [P10] for $\delta(\mathcal{C})$-regular σ-additive vector measures on $\delta(\mathcal{C})$.

Theorem 22.5. Let X be a quasicomplete lcHs and let $\mathbf{m} : \delta(\mathcal{C}) \to X$ be σ-additive and $\delta(\mathcal{C})$-regular. Let $f : T \to \boldsymbol{K}$ Then f is $\mathbf{m}$-integrable in T if and only if $f \in \mathcal{L}_1(x^* \circ \mathbf{m})$ for each $x^* \in X^*$ and, for each open Baire set U in T, there exists a vector $x_U \in X$ such that

$$x^*(x_U) = \int_U f d(x^* \circ \mathbf{m}) \qquad (22.5.1)$$

for $x^* \in X^*$.

Proof. Clearly, the conditions are necessary. Conversely, let the conditions hold. For each $q \in \Gamma$, $\Pi_q : X \to X_q \subset \widetilde{X_q}$ is continuous. Hence $(y^* \circ \Pi_q) \in X^*$ for $y^* \in X_q^*$ and hence by hypothesis $f \in \mathcal{L}_1(y^* \circ \mathbf{m}_q)$ for each $y^* \in X_q^*$ and by (22.5.1) we have

$$(y^* \circ \Pi_q)(x_U) = \int_U f d(y^* \circ \Pi_q \mathbf{m}) = \int_U f d(y^* \circ \mathbf{m}_q)$$

for each open Baire set U in T. Then by the last part of Theorem 22.4, $f \in \mathcal{L}_1(\mathbf{m}_q)$ for each $q \in \Gamma$. Particularly, f id $\mathbf{m}_q$-measurable for each $q \in \Gamma$ and hence f is $\mathbf{m}$-measurable. Moreover, by Definition 12.1 of [P10], $f \in \mathcal{L}_1(\mathbf{m})$ and

$$\int_A f d\mathbf{m} = \lim_{\longleftarrow} \int_A f d\mathbf{m}_q,\ A \in \mathcal{B}_c(T)$$

and

$$\int_T f d\mathbf{m} = \lim_{\longleftarrow} \int_{N(f)\setminus N_q} f d\mathbf{m}_q$$

(see Definition 12.1 of [P10]).

In order to generalize Theorem 22.4 to complete lcHs-valued σ-additive $\delta(\mathcal{C})$-regular vector measures on $\delta(\mathcal{C})$, we first generalize Lemmas 22.1 and 22.2 as follows.

Lemma 22.6. Let X be an lcHs with topology τ and let H be a subset of X^* such that τ is identical with the topology of uniform convergence in equicontinuous subsets of H. Let $\mathcal{P}$ be a δ-ring of subsets of a set $\Omega(\not\emptyset)$ and let $\mathbf{m} : \mathcal{P} \to X$ be additive. Let $\mathcal{E}_H = \{E \subset H : E \,\text{equicontinuous}\}$. Then:

(i) $||\mathbf{m}||_{q_E}(A) = \sup_{x^* \in E} v(x^* \circ \mathbf{m})(A)$, $A \in \sigma(\mathcal{P})$.

(ii) Suppose X is further quasicomplete, $\mathbf{m}$ is σ-additive and $f : \Omega \to \boldsymbol{K}$ is $\mathbf{m}$-measurable and $(x^* \circ \mathbf{m})$-integrable for each $x^* \in H$. Then $f \in \mathcal{L}_1(\mathbf{m})$ if, for each $E \in \mathcal{E}_H$ and $\epsilon > 0$, there exists $g_E^{(\epsilon)} \in \mathcal{L}_1(\mathbf{m}_{q_E})$ such that

$$\sup_{x^* \in E} \int_T |f - g_E^{(\epsilon)}| dv(x^* \circ \mathbf{m}) < \epsilon.$$

Proof. (i) is due to Proposition 10.12(iii) of [P10].

(ii) For $x^* \in E$, let Ψ_{x^*} be as in Proposition 10.12 of [P10]. Then by hypothesis and by the latter proposition, we have

$$\sup_{x^* \in E} \int_T |f - g_E^{(\epsilon)}| dv(\Psi_{x^*} \circ \mathbf{m}_{q_E}) = \sup_{x^* \in E} \int_T |f - g_E^{(\epsilon)}| dv(x^* \circ \mathbf{m}) < \epsilon. \quad (22.6.1)$$

Then by (22.6.1) and by Lemma 22.1(ii) applied to $\mathbf{m}_{q_E} : \mathcal{P} \to X_{q_E} \subset \widetilde{X_{q_E}}$, $f \in \mathcal{L}_1(\mathbf{m}_{q_E})$ since $\{\Psi_{x^*} : x^* \in E\}$ is a norm determining set for $\widetilde{X_{q_E}}$ by Proposition 10.12 of [P10] and since f is $\mathbf{m}_{q_E}$-measuable by hypothesis. Since E is arbitrary in $\mathcal{E}_H$ and since $\{q_E : E \in \mathcal{E}_H\}$ generates the topology τ, by Definition 12.1 and Remark 10.5 of [P12] we conclude that $f \in \mathcal{L}_1(\mathbf{m})$.

Lemma 22.7. Let X be an lcHs and let H satisfy the hypothesis of Lemma 22.6. Let $\mathcal{E}_H$ be as in the above lemma. Suppose $\mathbf{m} : \delta(\mathcal{C}) \to X$ (resp. $\mathbf{m} : \mathcal{B}(T) :\to X$) is σ-additive and let $V \in \mathcal{V}$. Then a set $A \in \mathcal{B}(V)$ (resp. $A \in \mathcal{B}(T)$) is $\mathbf{m}$-null if and only if A is $(x^* \circ \mathbf{m})$-null for each $x^* \in H$. If $\mathbf{m}$ is further $\delta(\mathcal{C})$-regular (resp. $\mathcal{B}(T)$-regular) and if $f : T \to \boldsymbol{K}$is $(x^* \circ \mathbf{m})$-measurable for $x^* \in H$, then f is Lusin $\mathbf{m}$-measurable as well as $\mathbf{m}$-measurable.

Proof. Let A be $(x^* \circ \mathbf{m})$-null for each $x^* \in H$ and let $E \in \mathcal{E}_H$. For $x^* \in E$, let Ψ_{x^*} be as in Proposition 10.12 of [P10]. Then by the said proposition $H_E = \{\Psi_{x^*} : x^* \in E\}$ is a norm determining set for X_{q_E} and hence for $\widetilde{X_{q_E}}$ and therefore, by hypothesis and by Lemma 22.2, A is $\mathbf{m}_{q_E}$-null. As E is arbitrary in $\mathcal{E}_H$ and as $\{q_E : E \in \mathcal{E}_H\}$ generates the topology τ, A is $\mathbf{m}$-null by Remark 10.5 of [P10]. Conversely, if A is $\mathbf{m}$-null, clearly A is $(x^* \circ \mathbf{m})$-null for each $x^* \in H$.

Let f be $(x^* \circ \mathbf{m})$-measurable for $x^* \in H$. Let $E \in \mathcal{E}_H$. Then, f is $(\Psi_{x^*} \circ \mathbf{m})$-measurable for $\Psi_{x^*} \in H_E$. Hence, by Lemma 22.2 applied to $\mathbf{m}_{q_E} : \mathcal{P} \to X_{q_E} \subset \widetilde{X_{q_E}}$, f is $\mathbf{m}_{q_E}$-measurable, where $\mathcal{P} = \delta(\mathcal{C})$ (resp. $\mathcal{B}(T)$. Then by Definition 10.6 and by Remark 10.5 of [P10], f is $\mathbf{m}$-measurable and consequently, f is also Lusin $\mathbf{m}$-measurable by Theorem 21.6 (resp. by Theorem 21.5).

Now we shall generalize Theorem 22.4 to complete lcHs-valued σ-additive $\delta(\mathcal{C})$-regular measures.

Theorem 22.8 (Generalization of Theorem 22.4 to complete lcHs-valued measures). Let X be a complete lcHs with topology τ and let H be a subset of X^* with the Orlicz property such that τ is identical with the toplogy of uniform convergence in equicontinuous subsets of H. Let $\mathbf{m} : \delta(\mathcal{C}) \to X$ be σ-additive and $\delta(\mathcal{C})$-regular and let $f : T \to \boldsymbol{K}$ Then $f \in \mathcal{L}_1(\mathbf{m})$ if and only if $f \in \mathcal{L}_1(x^* \circ \mathbf{m})$ for each $x^* \in H$ and, for each open Baire set U in T, there exists $x_U \in X$ such that

$$x^*(x_U) = \int_U f d(x^* \circ \mathbf{m}) \qquad (22.8.1)$$

for $x^* \in H$.

Proof. Let $\mathcal{E}_H = \{E \subset H : E \,\text{equicontinuous}\}$. By hypothesis, the seminorms q_E, $E \in \mathcal{E}_H$, generate the topology τ. Let $E \in \mathcal{E}_H$ be fixed. By Proposition 10.12 of [P10], $(\Psi_{x^*} \circ \Pi_{q_E})(x) = x^*(x)$ for $x \in X$ and for $x^* \in E$ and hence we identify $x^* \in E$ with Ψ_{x^*}. Let $H_E = U^0_{q_E} \cap H$ for $E \in \mathcal{E}_H$. Since $E \subset H_E \subset U^o_{q_E}$ and since $\{\Psi_{x^*} : x^* \in E\}$ is a norm determining set for X_{q_E} by the Proposition 10.12 of [P10], it follows that H_E is a norm determining set for X_{q_E} and hence for $\widetilde{X_{q_E}}$. Let $\mathcal{F}$ be the vector space spanned by the characteristic functions of open Baire sets in T. Then by (22.8.1), for each $g \in \mathcal{F}$, there exists $x_g \in X$ such that

$$x^*(x_g) = \int_T fg d(x^* \circ \mathbf{m}) \qquad (22.8.2)$$

for $x^* \in H$ and x_g is unique as τ is generated by $\{q_E : E \in \mathcal{E}_H\}$. Then the mapping $\Phi : \mathcal{F} \to X$ given by $\Phi(g) = x_g$ for $g \in \mathcal{F}$ is well defined and linear. For $E \in H_{\mathcal{E}}$, arguing as in the proof of Theorem 22.4 with $\widetilde{X_{q_E}}, \mathbf{m}_{q_E}$ and H_E in place of X, H and $\mathbf{m}$, respectively, and using (22.8.1) in place of (22.4.1) we can show that $\Pi_{q_E} \circ \Phi : \mathcal{F} \to \widetilde{X_{q_E}}$ is continuous for $E \in \mathcal{E}_H$. Therefore, there exists a unique continuous linear extension $\widehat{\Phi^{(E)}}$ of $\Pi_{q_E} \circ \Phi$ to the whole of $\bar{\mathcal{F}}$ with values in $\widetilde{X_{q_E}}$ where $\bar{\mathcal{F}}$ is the closure of $\mathcal{F}$ in the Banach space of all bounded scalar functions on T with supremum norm. Then there exists a constant M_E such that

$$|\widehat{\Phi^{(E)}}(\varphi)|_{q_E} \leq M_E ||\varphi||_T \qquad (22.8.3)$$

for $\varphi \in \bar{\mathcal{F}}$. Hence by Claim 3 in the proof of Theorem 22.3, $\widehat{\Phi_0^{(E)}} = \widehat{\Phi^{(E)}}|_{C_0(T)}$ is continuous and linear and has range in $\widetilde{X_{q_E}}$. Then (22.8.3) also holds for $\widehat{\Phi_0^{(E)}}$ with $\varphi \in C_0(T)$. Moreover, (22.8.3) holds for all $E \in \mathcal{E}_H$.

By hypothesis, X is a complete lcHs and $\{q_E : E \in \mathcal{E}_H\}$ generates the topology τ and hence by Theorem 5.4, §5, Ch. II of [Scha], $X = \varprojlim \widetilde{X_{q_E}}$. Let us define $\hat{\Phi} : \bar{\mathcal{F}} \to X$ by $\hat{\Phi}(\varphi) = \varprojlim \widehat{\Phi^{(E)}}(\varphi)$ for $\varphi \in \mathcal{F}$. Let $\Phi_0 : C_0(T) \to X$ be given by $\Phi_0 = \hat{\Phi}|_{C_0(T)}$. Then by (22.8.3) we have $|\hat{\Phi}(\varphi)|_{q_E} = |(\Pi_{q_E} \circ \Phi)(\varphi)|_{q_E} = |\widehat{\Phi^{(E)}}(\varphi)|_{q_E} \leq M_E||\varphi||_T$ for each $E \in \mathcal{E}_H$ and for each $\varphi \in \bar{\mathcal{F}}$. Hence $\hat{\Phi}$ and Φ_0 are X-valued continuous linear mappings.

Let $\boldsymbol{\eta}$ be the representing measure of Φ_0 in the sense of Definition 4 of [P5]. Then by Theorem 1 of [P5]

$$x^*\Phi_0(\varphi) = \int_T \varphi d(x^* \circ \boldsymbol{\eta}) \qquad (28.8.4)$$

for $x^* \in X^*$ and for $\varphi \in C_0(T)$.

Claim 1. $\int_T f\varphi d(x^* \circ \mathbf{m}) = x^*\Phi_0(\varphi) = \int_T \varphi d(x^* \circ \boldsymbol{\eta})$ for $x^* \in H$ and for $\varphi \in C_0(T)$.

In fact, let $\varphi \in C_0(T)$. As $f \in \mathcal{L}_1(x^* \circ \mathbf{m})$ for $x^* \in H$ and as φ is bounded and $\mathcal{B}_c(T)$-measurable, $f\varphi \in \mathcal{L}_1(x^* \circ \mathbf{m})$ for $x^* \in H$. By Claim 3 in the proof of Theorem 22.3, there exists $(g_n)_1^\infty \subset \mathcal{F}$ such that $||\varphi - g_n||_T \to 0$. Then, for $x^* \in H$,

$$|\int_T f\varphi d(x^* \circ \mathbf{m}) - \int_T fg_n d(x^* \circ \mathbf{m})| \leq ||\varphi - g_n||_T \int_T |f| dv(x^* \circ \mathbf{m}) \to 0 \qquad (22.8.5)$$

as $n \to \infty$. Observing that $\hat{\Phi}(g) = \Phi(g) = x_g$ for $g \in \mathcal{F}$, by (22.8.2), (22.8.4) and (22.8.5) and by the fact that $||\varphi - g_n||_T \to 0$ as $n \to \infty$ we have

$$\int_T f\varphi d(x^* \circ \mathbf{m}) = \lim_n \int_T fg_n d(x^* \circ \mathbf{m}) = \lim_n x^*(x_{g_n}) = \lim_n x^*\Phi(g_n) = x^*\Phi_0(\varphi) = \int_T \varphi d(x^* \circ \boldsymbol{\eta})$$

for $x^* \in H$ since Φ_0 is continuous on $C_0(T)$. Hence the claim holds.

Let U be an open Baire set in T and choose $(\varphi_n)_1^\infty \subset C_0(T)$ such that $\varphi_n \nearrow \chi_U$ (see the proof (22.4.13)). Then by Claim 1, by LDCT and by (22.8.1) we have

$$\begin{aligned}(x^* \circ \boldsymbol{\eta})(U) &= \lim_n \int_T \varphi_n d(x^* \circ \boldsymbol{\eta}) \\ &= \lim_n x^*\Phi_0(\varphi_n) = \lim_n \int_T f\varphi_n d(x^* \circ \mathbf{m}) \\ &= \int_T \chi_U f d(x^* \circ \mathbf{m}) = x^*(x_U) \qquad (22.8.6)\end{aligned}$$

for $x^* \in H$. Since H has the Orlicz property, by (22.8.6) and by Theorem 19.7(ii) of [P11], Φ_0 is weakly compact.

Then by Theorems 2 and 6 of [P5], $\boldsymbol{\eta}$ is X-valued, σ-additive and $\mathcal{B}(T)$-regular. Let $\nu_{x^*}(\cdot) = \int_{(\cdot)} fd(x^* \circ \mathbf{m})$ for $x^* \in H$. Then for $\varphi \in C_0(T)$ and for $x^* \in H$, by Claim 1 we have

$$\int_T \varphi d\nu_{x^*} = \int_T f\varphi d(x^* \circ \mathbf{m}) = x^*\Phi_0(\varphi) = \int_T \varphi d(x^* \circ \boldsymbol{\eta})$$

and hence an argument similar to to that in the paragraph following (22.4.10) in the proof of Theorem 22.4 shows that f is $(x^* \circ \boldsymbol{\eta})$-measurable for $x^* \in H$. Then by the last part of Lemma 22.7, f is $\boldsymbol{\eta}$-measurable as well as Lusin $\boldsymbol{\eta}$-measurable. Thus, given $E \in \mathcal{E}_H$ and $\epsilon > 0$, arguing as in the proof of Theorem 22.4, there exists a compact K_0 in T such that $||\boldsymbol{\eta}||_{q_E}(T\backslash K_0) < \epsilon$ and $f|_{K_0}$ is continuous. Then $f\chi_{K_0}$ is bounded and $\mathcal{B}_c(T)$-measurable with compact support. Consequently, by Theorem 11.9(i)(b) and by Notation 15.9 of [P10], $f\chi_{K_0} \in \mathcal{L}_1(\mathbf{m})$ and hence $f\chi_{K_0} \in \mathcal{L}_1(\mathbf{m}_{q_E})$. Moreover, by hypothesis f is $(x^* \circ \mathbf{m})$-measurable for each $x^* \in H$ and as H satisfies the hypothesis of Lemma 22.7, it follows that f is $\mathbf{m}$-measurable. Then an argument similar to that in the last part of the proof of Theorem 22.4, invoking Lemma 22.6(ii) in place of Lemma 22.1(ii), shows that $f \in \mathcal{L}_1(\mathbf{m}_{q_E})$. Since E is arbitrary in $\mathcal{E}_H$ and since $(q_E)_{E\in\mathcal{E}_H}$ generates the topology τ, by Definition 12.1 and by Remark 10.5 of [P10], $f \in \mathcal{L}_1(\mathbf{m})$.

This completes the proof of the theorem.

Now we give an analogue of Theorem 22.3 when $\mu_i : \mathcal{B}(T) \to \boldsymbol{K}$, $i \in I$, are σ-additive and Borel regular.

Theorem 22.9. Let $\mu_i : \mathcal{B}(T) \to \boldsymbol{K}$ be σ-additive and $\mathcal{B}(T)$-regular for $i \in I$. Suppose $\sum_{i\in I} |\int_T \varphi d\mu_i|^p < \infty$ for each $\varphi \in C_0(T)$ and for $1 \le p < \infty$. Let $u : C_0(T) \to \ell_p(I)$ be defined by $u(\varphi) = (\int_T \varphi d\mu_i)_{i\in I}$. Then u is a weakly compact operator on $C_0(T)$. Let $\mathbf{m}_u$ be the representing measure of u in the sense of Definition 4 of [P5] and let $f : T \to \boldsymbol{K}$ belong to $\mathcal{L}_1(\mu_i)$ for $i \in I$. Then f is $\mathbf{m}_u$-integrable in T if and only if

$$\sum_{i\in I} |\int_U f d\mu_i|^p < \infty \qquad (22.9.1)$$

for each open Baire set U in T. In that case, $\int_T f d\mathbf{m}_u = (\int_T f d\mu_i)_{i\in I}$.

Let $p = 1$ and let $f \in \mathcal{L}_1(\mathbf{m}_u)$. If $\theta(\varphi) = \sum_{i\in I} \int_T \varphi d\mu_i$ for $\varphi \in C_0(T)$, then $\theta \in \mathcal{K}(T)^*_b$ (see pp. 65 and 69 of [P2]), f is μ_θ-integrable and

$$\int_A f d\mu_\theta = \sum_{i\in I} \int_A f d\mu_i$$

for $A \in \mathcal{B}(T)$, where μ_θ is the bounded complex Radon measure induced by θ in the sense of Definition 4.3 of [P1].

Proof. By an argument similar to that in the proof of Theorem 22.3 we can show that the linear mapping u has closed graph and hence by Theorem 2.15 of [Ru2], u is continuous. Since $c_0 \not\subset \ell_p(I)$ for $1 \le p < \infty$, by Theorem 13 of [P5] or by Corollary 2 of [P6], u is weakly compact. Then by Theorems 2 and 6 of [P5], the representing measure $\mathbf{m}_u$ of u is $\ell_p(I)$-valued, σ-additive and Borel regular on $\mathcal{B}(T)$.

Let $H_I^{(p)}$ be as in the proof of Theorem 2.3 for $1 \le p < \infty$. If $x^* = (\alpha_i) \in H_I^{(p)}$, then there exists $J_{x^*} \subset I$, J_{x^*} finite, such that $\alpha_i = 0$ for $i \in I\backslash J_{x^*}$. Then by Theorem 1 of [P5] we have

$x^*u(\varphi) = \int_T \varphi d(x^* \circ \mathbf{m}_u) = \int_T \varphi d(\sum_{i \in J_{x^*}} \alpha_i \mu_i)$ for $\varphi \in C_0(T)$. Since $x^* \circ \mathbf{m}_u$ and μ_i, $i \in I$, are σ-additive and Borel regular, by the uniqueness part of the Riesz representation theorem we conclude that

$$x^* \circ \mathbf{m}_u = \sum_{i \in J_{x^*}} \alpha_i \mu_i = \sum_{i \in I} \alpha_i \mu_i. \qquad (22.9.2)$$

For $i \in I$, let θ_i and η_i be defined as in the proof of Theorem 22.3. Note that η_i is σ-additive on $\mathcal{B}(T)$ and $\eta_i \ll v(\mu_i)$, $i \in I$ so that η_i is $\mathcal{B}(T)$-regular for $i \in I$. Since

$$\int_T \varphi d\eta_i = \int_T \varphi f d\mu_i = \theta_i(\varphi) = \int_T \varphi d\mu_{\theta_i}$$

for $\varphi \in C_0(T)$ and for $i \in I$, by the uniqueness part of the Riesz representation theorem

$$\eta_i = \mu_{\theta_i} \quad \text{for } i \in I. \qquad (22.9.3)$$

Defining Ψ_{x^*} as in the proof of Theorem 22.3, by (22.9.3) we have

$$\mu_{\Psi_{x^*}} = \sum_{i \in J_{x^*}} \alpha_i \mu_{\theta_i} = \sum_{i \in J_{x^*}} \alpha_i \eta_i \text{ on } \mathcal{B}(T). \qquad (22.9.4)$$

Using (22.9.1), (22.9.4) and Hölder's inequality and arguing as in the proof of Claim 2 in the proof of Theorem 22.3 one can show that

$$\sup_{x^* \in H_I^{(p)}} |\mu_{\Psi_{x^*}}(U)| \leq (\sum_{i \in I} |\int_U f d\mu_i|^p)^{\frac{1}{p}} < \infty \qquad (22.9.5)$$

for $1 \leq p < \infty$ and for an open Baire set U in T. As $\{\mu_{\Psi_{x^*}} : x^* \in H_I^{(p)}\} \subset M(T)$ for $1 \leq p < \infty$, by (22.9.5) and by Corollary 18.5 of [P11] we have

$$\sup_{x^* \in H_I^{(p)}} v(\mu_{\Psi_{x^*}}, \mathcal{B}(T))(T) = M_p \text{ (say) } < \infty \qquad (22.9.6)$$

for $1 \leq p < \infty$.

By an argument quite similar to the proof of Claim 4 in the proof of Theorem 22.3 and by the use of (22.9.6) in place of Claim 2 in the proof of the said theorem, we have

$$\sum_{i \in I} |\int_T f \varphi d\mu_i|^p < \infty \qquad (22.9.7)$$

for $\varphi \in C_0(T)$ and for $1 \leq p < \infty$.

Then the mapping $\xi : C_0(T) \to \ell_p(I)$ given by

$$\xi(\varphi) = (\int_T f \varphi d\mu_i)_{i \in I} = (\int_T \varphi d\eta_i)_{i \in I}$$

is well defined and lnear and has closed graph. Then by the closed graph theorem, ξ is continuous. Since $c_0 \not\subset \ell_p(I)$ for $1 \leq p < \infty$, by Theorem 13 of [P5], ξ is weakly compact. . Then by Theorems 2 and 6 of [P5], the representing measure $\mathbf{m}_\xi$ of ξ is $\ell_p(I)$-valued, σ-additive and Borel regular on $\mathcal{B}(T)$.

Then arguing as in the proof of Claim 5 in the proof of Theorem 22.3 and using the uniqueness part of the Riesz representation theorem (Borel version), we conclude that

$$(x^* \circ \mathbf{m}_\xi) = \sum_{i \in J_{x^*}} \alpha_i \eta_i = \sum_{i \in I} \alpha_i \eta_i \qquad (22.9.8)$$

for each $x^* \in H_I^{(p)}$, $1 \leq p < \infty$. Invoking the Borel case of Lemma 22.2 and using (22.9.8) we conclude that f is $\mathbf{m}$-measurable as well as Lusin $\mathbf{m}$-measurable. The rest of the argument in the proof of Theorem 22.3 holds here verbatim with $\mathcal{B}(T)$ in place of $\mathcal{B}_c(T)$ and $\mathcal{B}(T)$-regular in place of $\delta(\mathcal{C})$-regular.

This completes the proof of the theorem.

Using the above theorem for $I = \mathbb{N}$ and for $p = 1$, we obtain below the analogue of Theorem 22.4 for a Banach space-valued σ-additive Borel regular measure.

Theorem 22.10. Let X be a Banach space and let $\mathbf{m} : \mathcal{B}(T) \to X$ be σ-additive and Borel regular. Let H be a norm determining set for X with the Orlicz property and let $f : T \to \mathbb{K}$ Then the conclusions of Theorem 22.4 hold. (By Theorem 20.12, $\mathbf{m}|_{\delta(\mathcal{C})}$ satisfies the hypothesis of Theorem 22.4 and hence the conclusions of Theorem 22.4 hold for $\mathbf{m}|_{\delta(\mathcal{C})}$ but it requires a proof to show that they hold for $\mathbf{m}$ itself.)

Proof. By Theorem 4.2 of [P8], the conditions are necessary. Conversely, let $< H >$ and F be as in the proof of Theorem 22.4. If $x^* \in F$ and $x^* = \sum_1^\infty x_n^*$ with $(x_n^*) \subset H$ and with $\sum_1^\infty |x_n^*| < \infty$, then arguing as in the proof of the said theorem, we have $x^* \circ \mathbf{m} = \sum_1^\infty x_n^* \circ \mathbf{m}$ and moreover, $x^* \circ \mathbf{m}$ is σ-additive and $\mathcal{B}(T)$-regular. Further, (22.4.3) holds for $\varphi \in C_0(T)$ and (22.4.4) also holds. Consequently, the mapping $u : C_0(T) \to \ell_1(\mathbb{N})$ given by

$$u(\varphi) = (\int_T \varphi d(x_n^* \circ \mathbf{m}))_{n=1}^\infty$$

is well defined and linear. Then the complex measures $(x_n^* \circ \mathbf{m})_{n=1}^\infty$ satisfy the hypotheses of Theorem 22.9 for $p = 1$ and for $I = \mathbb{N}$ and hence u is a weakly compact operator on $C_0(T)$ and $f \in \mathcal{L}_1(\mathbf{m}_u)$, where $\mathbf{m}_u$ is the representing measure of u. Then by the last part of the said theorem, $\theta : C_0(T) \to \mathbb{K}$ given by $\theta(\varphi) = \sum_{n=1}^\infty \int_T \varphi d(x_n^* \circ \mathbf{m})$, belongs to $\mathcal{K}(T)_b^*$, f is μ_θ-integrable and

$$\int_U f d\mu_\theta = \sum_{n=1}^\infty \int_U f d(x_n^* \circ \mathbf{m})$$

for each open Baire set U in T and for the set $U = N(f)$ since $f \in \mathcal{L}_1(\mathbf{m}_u)$ so that $N(f) \in \widetilde{\mathcal{B}(T)}$ with respect to $||\mathbf{m}_u||$. Observing that $x^* \circ \mathbf{m}$ is σ-additive and $\mathcal{B}(T)$-regular by hypothesis and μ_θ

is σ-additive and $\mathcal{B}(T)$-regular by Theorem 5.3 of [P2], and proving that $\int_T \varphi d\mu_\theta = \int_T \varphi d(x^* \circ \mathbf{m})$ for $\varphi \in C_0(T)$ as in the proof of Claim 1 in the proof of Theorem 22.4, by the uniqueness part of the Riesz representation theorem we conclude that $\mu_\theta = x^* \circ \mathbf{m}$. Moreover, by (22.4.2) and (22.4.5) we have $x^*(x_U) = \int_U f d(x^* \circ \mathbf{m})$, where by hypothsis $y^*(x_U) = \int_T f d(y^* \circ \mathbf{m})$ for each open Baire set U in T and for each $y^* \in H$.

Arguing as in the proof of Theorem 22.4, one can define the continuous linear mapping $\Phi_0 : C_0(T) \to X$ with the representing measure $\boldsymbol{\eta}$. Let $\nu_{x^*}(\cdot) = \int_{(\cdot)} f d(x^* \circ \mathbf{m})$ for $x^* \in H$. Then $\nu_{x^*} \ll v(x^* \circ \mathbf{m})$. Then f is ν_{x^*}-measurable since f is $(x^* \circ \mathbf{m})$-measurable and ν_{x^*} is $\mathcal{B}(T)$-regular for $x^* \in H$. Arguing as in the proof of Theorem 22.4, by the uniqueness part of the Riesz representation theorem we have $\nu_{x^*} = x^* \circ \boldsymbol{\eta}$, $x^* \in H$ and hence f is $(x^* \circ \boldsymbol{\eta})$-measurable for $x^* \in H$. The remaining arguments in the proof of Theorem 22.4 hold here excepting that the Borel case of Lemma 22.2 has to be invoked here.

This completes the proof of the theorem.

The following theorem which improves Theorem 12.2(i) of [P10] for $\mathcal{B}(T)$-regular σ-additive vector measures is immediate from the last part of Theorem 22.10 by an argument similar to that in the proof of Theorem 22.5.

Theorem 22.11. Let X be a quasicomplete lcHs and let $\mathbf{m} : \mathcal{B}(T) \to X$ be σ-additive and $\mathcal{B}(T)$-regular. Then a function $f : T \to \boldsymbol{K}$ is $\mathbf{m}$-integrable in T if and only if $f \in \mathcal{L}_1(x^* \circ \mathbf{m})$ for each $x^* \in X^*$ and, for each open Baire set U in T, there exists a vector $x_U \in X$ such that

$$x^*(x_U) = \int_U f d(x^* \circ \mathbf{m})$$

for $x^* \in X^*$.

The following result is an analogue of Theorem 22.8 for the Borel-regular σ-additive X-valued vector measure $\mathbf{m}$ where X is a complete lcHs.

Theorem 22.12. Let X, τ and H be as in Theorem 22.8 and let $\mathbf{m} : \mathcal{B}(T) \to X$ be σ-additive and Borel regular. Then a function $f : T \to \boldsymbol{K}$ belongs to $\mathcal{L}_1(\mathbf{m})$ if and only if $f \in \mathcal{L}_1(x^* \circ \mathbf{m})$ for each $x^* \in H$ and, for each open Baire set U in T, there exists a vector $x_U \in X$ such that

$$x^*(x_U) = \int_U f d(x^* \circ \mathbf{m})$$

for $x^* \in H$.

Proof. The proof of Theorem 22.8 holds here verbatim excepting that we have to invoke the uniqueness part of the Riesz representation theoem (Borel version) to show that $\nu_{x^*} = x^* \circ \boldsymbol{\eta}$ for $x^* \in H$ so that f is $(x^* \circ \boldsymbol{\eta})$-measurable for $x^* \in H$ and then invoke the Borel case of Lemma

22.7. The details are left to the reader.

23. ADDITIONAL CONVERGENCE THEOREMS

First we obtain a generalization of the Bourbaki version of the Egoroff theorem for an lcHs-valued σ-additive $\delta(\mathcal{C})$-regular measure. Then Proposition 4.3 of [T] is suitably generalized in Theorem 23.4. Corollary T_2 on P. 176 of [T] is improved in Theorem 23.6. Then the latter theorem is generalized to vector measures in Theorems 23.8 and 23.12.

Theorem 23.1 (Generalization of the Bourbaki version of the Egoroff theorem). Let X be an lcHs, $\mathbf{n}: \delta(\mathcal{C}) \to X$ be σ-additive and $\delta(\mathcal{C})$-regular and $f_0 : T \to \mathbb{K}$. For each $q \in \Gamma$, let $f_n^{(q)} : T \to \mathbb{K}$ be $\mathbf{n}_q$-measurable for $n \in \mathbb{N}$ and let $f_n^{(q)} \to f_0$ $\mathbf{n}_q$-a.e. in T. Then:

(i) f_0 is $\mathbf{n}$-measurable.

(ii) Given $K \in \mathcal{C}$, $q \in \Gamma$ and $\epsilon > 0$, there exists $K_0^{(q)} \in \mathcal{C}$ such that $K_0^{(q)} \subset K$, $||\mathbf{n}||_q(K\backslash K_0^{(q)}) < \epsilon$, $f_n^{(q)}|_{K_0^{(q)}}$, $n \in \mathbb{N} \cup \{0\}$, are continuous and $f_n^{(q)} \to f_0$ uniformly in $K_0^{(q)}$.

Proof. (i) By hypothesis, f_0 is $\mathbf{n}_q$-measurable for each $q \in \Gamma$ and hence f_0 is $\mathbf{n}$-measurable.

(ii) Without loss of generality we shall assume X to be a normed space. Clearly, $\delta(\mathcal{C}) \cap K = \mathcal{B}(K)$ is a σ-algebra and hence by Theorem 5.18(viii) of [P9], given $\epsilon > 0$, there exists $A_\epsilon \in \mathcal{B}(K)$ such that $||\mathbf{n}||(A_\epsilon) < \frac{\epsilon}{3}$ and $f_n \to f$ uniformly in $K\backslash A_\epsilon$. As $K\backslash A_\epsilon \in \mathcal{B}(K) \subset \delta(\mathcal{C})$, by the $\delta(\mathcal{C})$-regularity of $\mathbf{n}$ there exists a compact set $K_1 \subset K\backslash A_\epsilon$ such that $||\mathbf{n}||(K\backslash A_\epsilon\backslash K_1) < \frac{\epsilon}{3}$. Then particularly, $f_n \to f_0$ uniformly in K_1. Moreover, by hypothesis, by Theorem 21.6 and by Definition 21.3, for each n there exists $C_n \in \mathcal{C}$ with $C_n \subset K_1$ such that $f_n|_{C_n}$ is continuous and $||\mathbf{n}||(K_1\backslash C_n) < \frac{\epsilon}{3} \cdot \frac{1}{2^n}$. Then $K_0 = \bigcap_1^\infty C_n \in \mathcal{C}$, $K_0 \subset K_1 \subset K$ and $f_n|_{K_0}$ is continuous for each $n \in \mathbb{N}$ so that their uniform limit f_0 is also continuous in K_0. Moreover, $||\mathbf{n}||(K_1\backslash K_0) \leq \sum_{n=1}^\infty ||\mathbf{n}||(K_1\backslash C_n) < \frac{\epsilon}{3}$ so that $||\mathbf{n}||(K\backslash K_0) < \epsilon$.

The following definition is motivated by that on p.122 of [T].

Definition 23.2. Let X be an lcHs, $\mathbf{n} : \delta(\mathcal{C}) \to X$ be σ-additive, $f : T \to \mathbb{K}$ and $q \in \Gamma$. A sequence $(f_n^{(q)})$ of $\mathbf{n}_q$-measurable scalar functions is said to converge to f in measure $\mathbf{n}_q$ over compacts, if, given $K \in \mathcal{C}$ and $\eta > 0$, the sequence $||\mathbf{n}||_q(\{t \in K : |f_n^{(q)}(t) - f(t)| \geq \eta\}) \to 0$ as $n \to \infty$.

Theorem 23.3. Let X be an lcHs, $\mathbf{n} : \delta(\mathcal{C}) \to X$ be σ-additive, $q \in \Gamma$ and f_0, g_0, g, $f_n^{(q)}$, $g_n^{(q)}$: $T \to \mathbb{K}$, $n \in \mathbb{N}$, be $\mathbf{n}_q$-measurable for $n \in \mathbb{N}$. Let $f : T \to \mathbb{K}$. If $f_n^{(q)} \to f_0$ and $g_n^{(q)} \to g_0$ in measure $\mathbf{n}_q$ over compacts, then the following hold:

(i) $f_n^{(q)} + g_n^{(q)} \to f_0 + g_0$ in measure $\mathbf{n}_q$ over compacts.

(ii) $\lambda f_n^{(q)} \to \lambda f_0$ in measure $\mathbf{n}_q$ over compacts.

(iii) If $f_n^{(q)} \to g$ in measure $\mathbf{n}_q$ over compacts, then $f_0 = g$ $\mathbf{n}_q$-a.e. in T.

(iv) If $f_n^{(q)} \to f$ $\mathbf{n}_q$-a.e. in T and if $\mathbf{n}$ is $\delta(\mathcal{C})$-regular, then $f_n^{(q)} \to f$ in measure $\mathbf{n}_q$ over compacts.

Proof. (ii) is obvious. Let $K \in \mathcal{C}$ and $\eta > 0$. For any three $\mathbf{n}_q$-measurable scalar functions $\varphi, \psi,$ and h on T, it is easy to verify that $||\mathbf{n}||_q(\{t \in K : |\varphi(t) - \psi(t)| \geq \eta\}) \leq ||\mathbf{n}||_q(\{t \in K : |\varphi(t) - h(t)| \geq \frac{\eta}{2}\}) + ||\mathbf{n}||_q(\{t \in K : |h(t) - \psi(t)| \geq \frac{\eta}{2}\})$. Using this inequality, one can prove (i) and that $||\mathbf{n}||_q(N(f_0 - g) \cap K) = 0$ in (iii) for each $K \in \mathcal{C}$, since $N(f_0 - g) \cap K = \bigcup_{n=1}^{\infty}\{t \in K : |f_0(t) - g(t)| \geq \frac{1}{n}\}$. If $A = N(f_0 - g)$, then $A \in \widetilde{\mathcal{B}_c(T)}_q$ so that A is of the form $A = B \cup N$, $B \in \mathcal{B}_c(T)$, $N \subset M \in \mathcal{B}_c(T)$ and $||\mathbf{n}||_q(M) = 0$. Then there exists an increasing sequence $(B_n)_1^\infty \subset \delta(\mathcal{C})$ such that $B_n \nearrow B$. Then $K_n = \bar{B}_n \in \mathcal{C}$ and $B \subset \bigcup_1^\infty K_n$. Hence $||\mathbf{n}||_q(B) \leq \sum_1^\infty ||\mathbf{n}||_q(N(f_0 - g) \cap K_n) = 0$. Hence $f_0 = g$ $\mathbf{n}_q$-a.e. in T.

(iv) By hypothesis, f is $\mathbf{n}_q$-measurable. Let $K \in \mathcal{C}$ and $\eta > 0$. By hypothesis and by Theorem 23.1 there exists $K_0^{(q)} \in \mathcal{C}$ with $K_0^{(q)} \subset K$ such that $||\mathbf{n}||_q(K \backslash K_0^{(q)}) < \eta$, $f_n^{(q)}|_{K_0^{(q)}}$, $n \in \mathbb{N}$ and $f|_{K_0^{(q)}}$ are continuous and $f_n^{(q)} \to f$ uniformly in $K_0^{(q)}$. Hence, given $\epsilon > 0$, there exists n_0 such that $\sup_{t \in K_0^{(q)}} |f_n^{(q)}(t) - f(t)| < \epsilon$ for $n \geq n_0$ so that $||\mathbf{n}||_q(\{t \in K : |f_n^{(q)}(t) - f(t)| \geq \epsilon\}) \leq ||\mathbf{n}||_q(K \backslash K_0^{(q)}) < \eta$ for $n \geq n_0$. Hence (iv) holds.

Theorem 23.4 (A variant of Theorem 15.12(ii) of [P10]). Let X be a quasicomplete (resp. sequentially complete) lcHs, let $1 \leq p < \infty$ and let $\mathbf{n} : \delta(\mathcal{C}) \to X$ be σ-additive. Let $(f_n^{(q)})_1^\infty \subset \mathcal{L}_p(\mathbf{n}_q)$ (resp. $\subset \mathcal{L}_p(\sigma(\delta(\mathcal{C})), \mathbf{n}_q)$) for each $q \in \Gamma$ and let $f : T \to \mathbb{K}$ (resp. be $\mathcal{B}_c(T)$-measurable). Suppose $f_n^{(q)} \to f$ $\mathbf{n}_q$-a.e. in T for each $q \in \Gamma$. Then $f \in \mathcal{L}_p(\mathbf{n})$ (resp. $f \in \mathcal{L}_p(\sigma(\delta(\mathcal{C})), \mathbf{n})$) and $\lim_n (\mathbf{n}_q)_p^\bullet(f_n^{(q)} - f, T) = 0$ for each $q \in \Gamma$ if and only if the following conditions hold:

(i) $(\mathbf{n}_q)_p^\bullet(f_n^{(q)}, \cdot)$, $n \in \mathbb{N}$, are uniformly $\mathbf{n}_q$-continuous on $\mathcal{B}_c(T)$ for each $q \in \Gamma$. (See Definition 8.3 of [P10].)

(ii) For each $\epsilon > 0$ and $q \in \Gamma$, there exists $K^{(q)} \in \mathcal{C}$ such that

$$\sup_n (\mathbf{n}_q)_p^\bullet(f_n^{(q)}, T \backslash K^{(q)}) < \epsilon.$$

Proof. If $\mathcal{P} = \delta(\mathcal{C})$, note that $\sigma(\mathcal{P}) = \mathcal{B}_c(T)$. Then (i) is the same as condition (a) of Theorem 15.12(ii) of [P10]. (ii) is equivalent to condition (b) of Theorem 15.2(ii), since for $A_\epsilon^{(q)} \in \mathcal{P}$, $\overline{A_\epsilon^{(q)}} = K^{(q)} \in \mathcal{C}$ and $T \backslash A_\epsilon^{(q)} \supset T \backslash K^{(q)}$.

Remark 23.5. In the light of Lemma 20.5 and Theorem 20.12, Theorems 23.1 and 23.3(iv) hold for $\mathbf{n} = \mathbf{m}|_{\delta(\mathcal{C})}$ (resp. $\mathbf{n} = \boldsymbol{\omega}|_{\delta(\mathcal{C})}$) where $\mathbf{m} : \mathcal{B}(T) \to X$ (resp. $\boldsymbol{\omega} : \mathcal{B}_c(T) \to X$) is σ-additive and Borel regular (resp. and σ-Borel regular).

The following theorem is an improved version of Corollary T_2 on p. 176 of [T].

Theorem 23.6. Let $\theta \in \mathcal{K}(T)^*$ and $\lambda = \mu*_{|\theta|}|_{\mathcal{B}(T)}$ (so that $\lambda = v(\mu_\theta, \mathcal{B}(T)) = \mu_{|\theta|}|_{\mathcal{B}(T)}$ by Theorems 4.7 and 4.11 of [P1]). Then $\lambda : \mathcal{B}(T) \to [0, \infty]$ is σ-additive and Radon-regular in the sense of Definition 3.3 of [P1]. Suppose $(f_n)_1^\infty \subset \mathcal{L}_1(\lambda)$ such that the sequence $(\int_U f_n d\mu_\theta)_1^\infty$ is convergent for each open Baire set U in T. Then there exists $f \in \mathcal{L}_1(\lambda)$ such that $\lim_n \int_A f_n d\mu_\theta = \int_A f d\mu_\theta$ for each Borel set A in T. Consequently,

$$\lim_n \int_T g f_n d\mu_\theta = \int_T g f d\mu_\theta \qquad (23.6.1)$$

for each λ-measurable bounded scalar function g on T and consequently, f is unique in $L_1(\lambda)$. If $f_n \to h$ λ-a.e. in T, then $f = h$ λ-a.e. in T and $\lim_n \int_A f_n d\mu_\theta = \int_A h d\mu_\theta$ for each $A \in \mathcal{B}(T)$, the convergence being uniform with respect to $A \in \mathcal{B}(T)$.

Proof. λ is σ-additive and Radon-regular by Theorem 2.2 of [P1]. For $f \in L_1(\lambda)$, let $\mu(\cdot) = \int_{(\cdot)} f d\mu_\theta$. Since $f \in L_1(\lambda)$, by Lemma 1, no. 6, § 5, Ch. IV of [B], there exist a sequence $(K_n)_1^\infty \subset \mathcal{C}$ and a λ-null set N such that $N(f) \subset N \cup \bigcup_1^\infty K_n$. Hence, for $A \in \mathcal{B}(T)$, let us define

$$\int_A f d\mu_\theta = \int_{\bigcup_1^\infty (A \cap K_n)} f d\mu_\theta \qquad (23.6.2)$$

which is well defined as $f \in \mathcal{L}_1(\mu_\theta)$. Then we note that $\mu : \mathcal{B}(T) \to \boldsymbol{K}$ given by $\mu(A) = \int_A f d\mu_\theta$ is well defined, σ-additive and Borel regular since $\mu \ll v(\mu_\theta)$ on $\mathcal{B}_c(T)$ and hence on $\mathcal{B}(T)$ by (23.6.2). (See Claim (*) in the proof of Theorem 22.3.) Therefore, $\mu \in M(T)$. Moreover, by Notation 18.1 of [P11], by Proposition 2.11 of [P8], by Theorems 4.7 and 4.11 of [P1] and by (23.6.2) we have $||\mu|| = v(\mu, \mathcal{B}(T))(T) = v(\mu, \mathcal{B}_c(T))(\bigcup_1^\infty K_n) = \int_{\bigcup_1^\infty K_n} |f| dv(\mu_\theta, \delta(\mathcal{C})) = \int_{\bigcup_1^\infty K_n} |f| d\mu_{|\theta|}|_{\delta(\mathcal{C})} = \int_{\bigcup_1^\infty K_n} |f| d\mu_{|\theta|}|_{\mathcal{B}(T)} = \int_T |f| d\lambda = ||f||_1$. Then, the mapping $\Phi : L_1(\lambda) \to M(T)$ given by $\Phi(f)(\cdot) = \int_{(\cdot)} f d\mu_\theta$ is linear and isometric so that $M_\lambda = \Phi(L_1(\lambda)) = \{\mu \in M(T) :$ there exists $f \in L_1(\lambda)$ such that $\mu(\cdot) = \int_{(\cdot)} f d\mu_\theta\}$ is complete with respect to the norm on $M(T)$. Therefore, M_λ is a closed subspace of $M(T)$. Then by the Hahn-Banach theorem, M_λ is a weakly closed subspace of $M(T)$.

Let $\mu_n(\cdot) = \int_{(\cdot)} f_n d\mu_\theta$, $n \in \mathbb{N}$. Then by the foregoing argument μ_n, $n \in \mathbb{N}$, belong to $M(T)$. By hypothesis, $\lim_n \mu_n(U) \in \boldsymbol{K}$ for each open Baire set U in T and hence by Theorem 18.6 of [P11] there exists $\mu_0 \in M(T)$ such that $\mu_n \to \mu_0$ weakly in $M(T)$. As $(\mu_n)_1^\infty \subset M_\lambda$ and as M_λ is weakly closed, $\mu_0 \in M_\lambda$ and hence there exists $f \in L_1(\lambda)$ such that $\mu_0(\cdot) = \int_{(\cdot)} f d\mu_\theta$ on $\mathcal{B}(T)$. Moreover, as $\mu_n \to \mu_0$ weakly in $M(T)$, $(\mu_n)_{n=0}^\infty$ is weakly bounded and hence by Theorem 3.18 of [Ru2], $\sup_{n \in \mathbb{N} \cup \{0\}} ||\mu_n|| = M < \infty$ and further, $\mu_n(A) \to \mu_0(A)$ for each $A \in \mathcal{B}(T)$.

Let g be a bounded Borel measurable function on T and let $\epsilon > 0$. Then there exists a $\mathcal{B}(T)$-simple function s such that $||g - s||_T < \frac{\epsilon}{3M}$. Let $s = \sum_{i=1}^r \alpha_i \chi_{A_i}$ with $(A_i)_1^r \subset \mathcal{B}(T)$. Choose n_0

such that

$$|\alpha_i||\mu_n(A_i) - \mu_0(A_i)| < \frac{\epsilon}{3r} \qquad (23.6.2)$$

for $i = 1, 2, ..., r$ and for $n \geq n_0$. Then,

$$|\int_T g f_n d\mu_\theta - \int_T s f_n d\mu_\theta| \leq ||g - s||_T (\int_T |f_n| d\lambda) = ||g - s||_T ||\mu_n|| < \frac{\epsilon}{3} \quad (23.6.3)$$

for all $n \in \mathbb{N}$ and

$$|\int_T g f d\mu_\theta - \int_T s f d\mu_\theta| \leq ||g - s||_T (\int_T |f| d\lambda) = ||g - s||_T ||\mu_0|| < \frac{\epsilon}{3}. \quad (23.6.4)$$

Then by (23.6.2), (23.6.3) and (23.6.4) we have

$$|\int_T g f_n d\mu_\theta - \int_T g f d\mu_\theta| < \epsilon$$

for $n \geq n_0$ and hence (23.6.1) holds for bounded Borel measurable functions g.

If g is bounded and λ-measurable, then there exists a λ-null set $N \in \mathcal{B}(T)$ such that $g\chi_{T\setminus N}$ is Borel measurable and hence (23.6.1) holds for bounded λ-measurable functions g. Moreover, since (23.6.1) implies that $f_n \to f$ weakly in $L_1(\lambda)$, f is unique in $L_1(\lambda)$.

If $f_n \to h$ λ-a.e. in T, then by (23.6.1) (with $g = \chi_A$, $A \in \mathcal{B}_c(T)$) and by Proposition 2.13 of [P8], $\lim_n \int_A f_n d\mu_\theta = \int_T h d\mu_\theta$ for each $A \in \mathcal{B}_c(T)$ and consequently, again by (23.6.1), $\int_A h d\mu_\theta = \int_A f d\mu_\theta$ for $A \in \mathcal{B}_c(T)$. Let $\nu(A) = \int_A (f - h) d\mu_\theta$ for $A \in \mathcal{B}_c(T)$. Then ν is a null measure on $\mathcal{B}_c(T)$ and hence $v(\nu)(N(f-h)) = \int_{N(f-h)} |f-h| dv(\mu_\theta, \delta(\mathcal{C})) = \int_{N(f-h)} |f-h| d\mu_{|\theta|} = \int_{N(f-h)} |f - h| d\lambda = 0$ by Proposition 2.11 of [P8] and by Theorems 4.7 and 4.11 of [P1]. Therefore, $f = h$ λ-a.e. in T. Since $\int_A f d\mu_\theta = \int_A h d\mu_\theta = \int_{\bigcup_1^\infty (A\cap K_n)} h d\mu_\theta$ for $A \in \mathcal{B}(T)$, where $N(f) \subset N \cup \bigcup_1^\infty K_n$, $(K_n)_1^\infty \subset \mathcal{C}$ and N is λ-null, the last part holds by the equivalence of (i) and (iii) of Proposition 2.13 of [P8].

The rest of the section is devoted to generalize Theorem 23.6 to vector measures. We begin with the following lemma.

Lemma 23.7. Let X be a quasicomplete lcHs and let $\mathbf{m} : \delta(\mathcal{C}) \to X$ be σ-additive and $\delta(\mathcal{C})$-regular. Let $f : T \to \mathbb{K}$ belong to $\mathcal{L}_1(\mathbf{m})$. Then:

(i) There exists $B \in \mathcal{B}_c(T)$ such that $N(f) \subset B$.

(ii) Let $\boldsymbol{\gamma} : \mathcal{B}(T) \to X$ be defined by $\boldsymbol{\gamma}(A) = \int_{A\cap B} f d\mathbf{m}$ for $A \in \mathcal{B}(T)$. Then $\boldsymbol{\gamma}$ is σ-additive and $\mathcal{B}(T)$-regular.

Proof. As f is $\mathbf{m}$-measurable, there exists $M \in \mathcal{B}_c(T)$ such that $||\mathbf{m}||(M) = 0$ and such that $f\chi_{T\setminus M}$ is $\mathcal{B}_c(T)$-measurable. Consequently, $N(f\chi_{T\setminus M}) \in \mathcal{B}_c(T)$ so that $N(f) \subset N(f\chi_{T\setminus M}) \cup$

$M = B$ (say). Thus (i) holds.

(ii) For $A \in \mathcal{B}(T)$, $A \cap B$ is σ-bounded and Borel and hence $A \cap B \in \mathcal{B}_c(T)$. Since $\int_{(\cdot)} f d\mathbf{m}$ is σ-additive on $\mathcal{B}_c(T)$ by Theorem 11.8(ii) and by Remark 12.5 of [P10], it follows that $\boldsymbol{\gamma}(\cdot) = \boldsymbol{\gamma}(\cdot \cap B)$ is σ-additive on $\mathcal{B}(T)$.

Claim 1. $\boldsymbol{\gamma}$ is Borel inner regular.

In fact, let $A \in \mathcal{B}(T)$ and let B be as in (i). Then there exists $(E_k)_1^\infty \subset \delta(\mathcal{C})$ such that $E_k \nearrow B \cap A$. Let $\epsilon > 0$ and $q \in \Gamma$. Since $\boldsymbol{\gamma}$ is σ-additive on $\mathcal{B}(T)$, by Proposition 2.2 of [P8], $||\boldsymbol{\gamma}||_q$ is continuous on $\mathcal{B}(T)$ and hence there exists k_0 such that $||\boldsymbol{\gamma}||_q((A \cap B)\backslash E_k) < \frac{\epsilon}{2}$ for $k \geq k_0$. On the other hand, by Theorem 11.8(iii)(c) and by Remark 12.5 of [P10], there exists $\delta > 0$ such that $||\boldsymbol{\gamma}||_q(F) < \frac{\epsilon}{2}$ whenever $F \in \mathcal{B}_c(T)$ with $||\mathbf{m}||_q(F) < \delta$. Since $\mathbf{m}$ is $\delta(\mathcal{C})$-regular by hypothesis, there exists $C \in \mathcal{C}$ such that $C \subset E_{k_0}$ and $||\mathbf{m}||_q(E_{k_0}\backslash C) < \delta$. Then $||\boldsymbol{\gamma}||_q(E_{k_0}\backslash C) < \frac{\epsilon}{2}$. Then $||\boldsymbol{\gamma}||_q(A\backslash C) < \epsilon$ since $||\boldsymbol{\gamma}||_q(A\backslash B) = 0$. Hence the claim holds.

$\Pi_q \circ \boldsymbol{\gamma} : \mathcal{B}(T) \to \widetilde{X_q}$ is σ-additive and hence has bounded range. Then by Proposition 10.14 of [P10], $\{x^* \circ \boldsymbol{\gamma} : x^* \in U_q^0\} = \{\Psi_{x^*}(\Pi_q \circ \boldsymbol{\gamma}) : x^* \in U_q^0\}$ is bounded in $M(T)$. By Claim 1 $\boldsymbol{\gamma}$ is Borel inner regular on $\mathcal{B}(T)$, and hence, given $A \in \mathcal{B}(T)$, $q \in \Gamma$ and $\epsilon > 0$, there exists a compact $K \subset A$ such that $||\boldsymbol{\gamma}||_q(A\backslash K) < \epsilon$. Then by the said proposition of [P10] we have

$$\sup_{x^* \in U_q^0} v(x^* \circ \boldsymbol{\gamma})(A\backslash K) = ||\Pi_q \circ \boldsymbol{\gamma}||(A\backslash K) = ||\boldsymbol{\gamma}||_q(A\backslash K) < \epsilon$$

and hence $\{(x^* \circ \boldsymbol{\gamma}) : x^* \in U_q^0\}$ is uniformly Borel inner regular on $\mathcal{B}(T)$. Consequently, by Theorem 2 of [P4], $\{x^* \circ \boldsymbol{\gamma} : x^* \in U_q^0\}$ is uniformly Borel regular on $\mathcal{B}(T)$ and arguing as in the above invoking Proposition 10.14 of [P10], we conclude that $\boldsymbol{\gamma}$ is Borel regular on $\mathcal{B}(T)$.

Using the above lemma, we give in the following theorem two generalizations of the a.e. convergence part of Theorem 23.6 to σ-additive $\delta(\mathcal{C})$-regular vector measures and this result is a strengthened vector measure analogue of Proposition 4.8 of [T].

Theorem 23.8. Let X be a quasicomplete lcHs with topology τ and let $\mathbf{m} : \delta(\mathcal{C}) \to X$ be σ-additive and $\delta(\mathcal{C})$-regular. Let $(f_n)_1^\infty \subset \mathcal{L}_1(\mathbf{m})$ and let $f : T \to \boldsymbol{K}$,or $[-\infty, \infty]$ be such that $f_n \to f$ $\mathbf{m}$-a.e. in T (see Definition 10.4 of [P10]).

(a) Suppose $(\int_U f_n d\mathbf{m})_1^\infty$ converges in X in τ for each open Baire set U in T. Then the following hold:

(a)(i) $f \in \mathcal{L}_1(\mathbf{m})$.

(a)(ii) For $A \in \mathcal{B}(T)$,

$$\int_A f_n d\mathbf{m} \to \int_A f d\mathbf{m} \quad \text{in } \tau$$

and for each $q \in \Gamma$, the onvergence is uniform with respect to $A \in \mathcal{B}(T)$.

(a)(iii) For bounded $\mathbf{m}$-measurable scalar functions g on T,

$$\int_T f_n g d\mathbf{m} \to \int_T f g d\mathbf{m} \quad \text{in } \tau.$$

(b) Suppose $(\int_U f_n d\mathbf{m})_1^\infty$ converges weakly in X for each open Baire set U in T. Then the following hold:

(b)(i) $f \in \mathcal{L}_1(\mathbf{m})$.

(b)(ii) For $A \in \mathcal{B}(T)$,

$$\int_A f_n d\mathbf{m} \to \int_A f d\mathbf{m} \text{ weakly.}$$

(b)(iii) For bounded $\mathbf{m}$-measurable scalar functions g on T

$$\int_T f_n g d\mathbf{m} \to \int_T f g d\mathbf{m} \text{ weakly.}$$

Proof.

(a)(i) and (b)(i). Let U be an open Baire set in T. By hypothesis (a) (resp. (b)) there exists a vector $x_U \in X$ such that

$$\lim_n \int_U f_n d\mathbf{m} = x_U \quad \text{in } \tau (\text{resp. weakly}).$$

Then in both the cases, by Theorem 11.8(v) and Remark 12.5 of [P10]

$$\lim_n \int_U f_n d(x^* \circ \mathbf{m}) = x^*(x_U) \qquad (23.8.1)$$

for $x^* \in X^*$. On the other hand, by hypothesis and by Theorem 4.4(i) of [P2], $x^* \circ \mathbf{m} = \mu_\theta$ for some $\theta \in \mathcal{K}(T)^*$ and hence by Theorem 23.6 we have

$$\lim_n \int_U f_n d(x^* \circ \mathbf{m}) = \int_U f d(x^* \circ \mathbf{m}) \qquad (23.8.2)$$

for $x^* \in X^*$. Then by (23.8.1) and (23.8.2) we have

$$x^*(x_U) = \int_U f d(x^* \circ \mathbf{m})$$

for each open Baire set U in T and for $x^* \in X^*$. Consequently, by Theorem 22.5, (a)(i) (resp. (b)(i)) holds.

(a)(ii) Let $f_0 = f$. By (a)(i) and by Lemma 23.7, there exists $B_n \in \mathcal{B}_c(T)$ such that $N(f_n) \subset B_n$ and $\boldsymbol{\gamma}_n : \mathcal{B}(T) \to X$ given by $\boldsymbol{\gamma}_n(A) = \int_{A \cap B_n} f_n d\mathbf{m}$ for $A \in \mathcal{B}(T)$, is σ-additive and Borel regular for $n \in \mathbb{N} \cup \{0\}$. By hypothesis, $\lim_n \boldsymbol{\gamma}_n(U) \in X$ in τ for each open Baire set U in T. Since X is also sequentially complete, by Theorem 18.23 of [P11] there exists a unique X-valued σ-additive measure $\boldsymbol{\gamma}$ on $\mathcal{B}(T)$ such that

$$\lim_n \int_T g d\boldsymbol{\gamma}_n = \int_T g d\boldsymbol{\gamma} \in X \qquad (23.8.3)$$

in τ for each bounded Borel measurable scalar function g on T.

Claim 1. For a bounded Borel measurable scalar function g on T

$$\int_T g d\boldsymbol{\gamma}_n = \int_T f_n g d\mathbf{m} \qquad (23.8.4)$$

for $n \in \mathbb{N}$.

In fact, (22.8.4) clearly holds for $\mathcal{B}(T)$-simple functions. Choose a sequence $(s_n)_1^\infty$ of $\mathcal{B}(T)$-simple functions such that $s_n \to g$ and $|s_n| \nearrow |g|$ pointwise (in fact, uniformly) in T. Then by LDCT (Theorem 15.3(i) of [P10]) and by the validity of (23.8.4) for $\mathcal{B}(T)$-simple functions we have

$$\int_T g d\boldsymbol{\gamma}_n = \lim_k \int_T s_k d\boldsymbol{\gamma}_n = \lim_k \int_T f_n s_k d\mathbf{m} = \int_T f_n g d\mathbf{m}$$

since g and $f_n g$ are $\mathbf{m}$-integrable in T and since g is $\boldsymbol{\gamma}_n$ integrable in T by (i)(b) and (ii) of Theorem 11.9 and by Remark 12.5 of [P10]. Hence the claim holds.

Let $B = \bigcup_{n=0}^\infty B_n$. Then $B \in \mathcal{B}_c(T)$ and $f_n = 0$ in $T \backslash B$ for $n \in \mathbb{N} \cup \{0\}$. For $x^* \in X^*$, let θ be as in the proof of (a)(i). Then by hypothesis (a), by Theorem 11.8(v) and Remark 12.5 of [P10], $(\int_U f_n d(x^* \circ \mathbf{m}))_1^\infty$ is convergent in $\mathbb{K}$ for each open Baire set U in T. Hence by Theorem 23.6,

$$\lim_n \int_T f_n g d(x^* \circ \mathbf{m}) = \int_T f g d(x^* \circ \mathbf{m}) \qquad (23.8.5)$$

for each bounded Borel measurable funxtion g on T. Then for $A \in \mathcal{B}(T)$, by (23.8.4) and (23.8.5) and by Theorem 11.8(v) and Remark 12.5 of [P10] (since $f \in \mathcal{L}_1(\mathbf{m})$ by (a)(i)), we have

$$\begin{aligned}\lim_n (x^* \circ \boldsymbol{\gamma}_n)(A) = \lim_n \int_A f_n d(x^* \circ \mathbf{m}) &= \lim_n \int_{A \cap B} f_n d(x^* \circ \mathbf{m}) \\ &= \int_{A \cap B} f d(x^* \circ \mathbf{m}) \\ &= \int_A f d(x^* \circ \mathbf{m}) \\ &= (x^* \circ \boldsymbol{\gamma}_0)(A). \quad (23.8.6)\end{aligned}$$

Moreover, by (23.8.3) we have

$$\lim_n (x^* \circ \boldsymbol{\gamma}_n)(A) = \lim_n (x^* \circ \boldsymbol{\gamma}_n)(A \cap B) = (x^* \circ \boldsymbol{\gamma})(A \cap B) = (x^* \circ \boldsymbol{\gamma})(A) \qquad (23.8.7)$$

for $x^* \in X^*$, since $\boldsymbol{\gamma}_n(A \backslash B) = 0$ for all n so that $\boldsymbol{\gamma}(A \backslash B) = 0$. Consequently, by (23.8.6) and (23.8.7) we have

$$(x^* \circ \boldsymbol{\gamma}_0)(A) = (x^* \circ \boldsymbol{\gamma})(A)$$

for $A \in \mathcal{B}(T)$ and for $x^* \in X^*$. Then, by the Hahn-Banach theorem, $\boldsymbol{\gamma}(A) = \boldsymbol{\gamma}_0(A)$ and hence, for $A \in \mathcal{B}(T)$, by (23.8.3) we have

$$\lim_n \int_A f_n d\mathbf{m} = \lim_n \boldsymbol{\gamma}_n(A) = \boldsymbol{\gamma}(A) = \boldsymbol{\gamma}_0(A) = \int_A f d\mathbf{m} \qquad (23.8.8)$$

in τ.

In Theorem 12.8 of [P10], take $f_n^{(q)} = f_n$ for all $q \in \Gamma$. Note that $\sigma(\delta(\mathcal{C})) = \mathcal{B}_c(T)$. Then in the notation of the said thorem, by Theorems 11.8(v) and 12.8 and by Remark 12.5 of [P10] and by (23.8.8) we have

$$\begin{aligned} \boldsymbol{\gamma}_n^{(q)}(A) = \boldsymbol{\gamma}_n^{(q)}(A \cap B) &= \int_{A \cap B} f_n d\mathbf{m}_q \\ &= \Pi_q\Big(\int_{A \cap B} f_n d\mathbf{m}\Big) \\ &\to (\Pi_q \circ \boldsymbol{\gamma}_0)(A \cap B) = (\Pi_q \circ \boldsymbol{\gamma}_0)(A) \end{aligned}$$

in $\widetilde{X_q}$ and the limit is uniform with resepect to $A \in \mathcal{B}(T)$ for a fixed $q \in \Gamma$, since it is so with respect to $E \in \mathcal{B}_c(T)$. Hence (a)(ii) holds.

(a)(iii) By hypothesis and by Theorem 18.8 of [P11], for each $q \in \Gamma$, there exists a finite constant M_q such that

$$\sup_{n \in \mathbb{N} \cup \{0\}} ||\boldsymbol{\gamma}_n||_q(T) = M_q. \qquad (23.8.9)$$

Let g be a bounded $\mathbf{m}$-measurable scalar function on T and let $q \in \Gamma$. Then there exists $N_q \in \mathcal{B}_c(T)$ with $||\mathbf{m}||_q(N_q) = 0$ such that $h_q = g\chi_{T \backslash N_q}$ is $\mathcal{B}_c(T)$-measurable and bounded. Hence, given $\epsilon > 0$, there exists a $\mathcal{B}_c(T)$-simple function $s^{(q)}$ such that $|s^{(q)}| \leq |h_q|$ and

$$||h_q - s^{(q)}||_T < \frac{\epsilon}{3M_q}. \qquad (23.8.10)$$

Let $s^{(q)} = \sum_{i=1}^r \alpha_i \chi_{A_i}$ with $(A_i)_1^r \subset \mathcal{B}_c(T)$. By (a)(ii) there exists n_1 such that

$$|\alpha_i||\boldsymbol{\gamma}_n(A_i) - \boldsymbol{\gamma}_0(A_i)|_q < \frac{\epsilon}{3r} \qquad (23.8.11)$$

for $i = 1, 2, ..., r$ and for $n \geq n_1$. Then by (23.8.11) we have

$$\Big|\int_T s^{(q)} d\boldsymbol{\gamma}_n - \int_T s^{(q)} d\boldsymbol{\gamma}_0\Big|_q \leq \sum_{i=1}^r |\alpha_i||\boldsymbol{\gamma}_n(A_i) - \boldsymbol{\gamma}_0(A_i)|_q < \frac{\epsilon}{3} \quad (23.8.12)$$

for $n \geq n_1$. Moreover, by (23.8.9) and (23.8.10) we have

$$|\int_T s^{(q)} d\boldsymbol{\gamma}_n - \int_T h_q d\boldsymbol{\gamma}_n|_q \leq ||s^{(q)} - h_q||_T ||\boldsymbol{\gamma}_n||_q(T) < \frac{\epsilon}{3} \quad (23.8.13)$$

for $n \in \mathbb{N} \cup \{0\}$.

Consequently, by (23.8.12) and (23.8.13)

$$\begin{aligned} |\int f_n g d\mathbf{m} - \int_T f g d\mathbf{m}|_q &= |\int_T f_n h_q d\mathbf{m} - \int_T f h_q d\mathbf{m}|_q \\ &\leq |\int_T (h_q - s^{(q)}) d\boldsymbol{\gamma}_n|_q + |\int_T s^{(q)} d\boldsymbol{\gamma}_n - \int_T s^{(q)} d\boldsymbol{\gamma}_0|_q \\ &+ |\int_T (s^{(q)} - h_q) d\boldsymbol{\gamma}_0|_q \\ &< \epsilon \end{aligned}$$

for $n \geq n_1$. Hence (a)(iii) holds.

(b)(ii) Let $x^* \in X^*$. Then by hypothesis and by Theorem 4.4(i) of [P2] there exists $\theta \in \mathcal{K}(T)^*$ such that $\mu_\theta = x^* \circ \mathbf{m}$. Then by hypothesis, by(b)(i) and by Theorem 23.6 and by Theorem 11.8(v) and Remark 12.5 of [P10], (b)(ii) holds.

(b)(iii) Let U be an open Baire set in T. Let $\boldsymbol{\gamma}_n$, $n \in \mathbb{N} \cup \{0\}$, be as in the proof of (a)(ii). By hypothesis, $\lim_n (x^* \circ \boldsymbol{\gamma}_n)(U) \in \boldsymbol{K}$ for each $x^* \in X^*$. Hence $((\boldsymbol{\gamma}_n)(U))_1^\infty$ is weakly bounded and hence, by Theorem 3.18 of [Ru2], is bounded in τ. Then for $q \in \Gamma$, by Theorem 18.8 of [P11], (23.8.9) holds.

Let $x^* \in X^*$ be fixed and let $q_{x^*}(x) = |x^*(x)|$ for $x \in X$. By hypothesis there exists a bounded $\mathcal{B}_c(T)$-measurable function h_{x^*} such that $h_{x^*} = g\chi_{T \setminus N_{x^*}}$ where $N_{x^*} \in \mathcal{B}_c(T)$ with $v(x^* \circ \mathbf{m})(N_{x^*}) = 0$. Choose a $\mathcal{B}_c(T)$-simple function s such that

$$||s - h_{x^*}||_T < \frac{\epsilon}{3M_{q_{x^*}}}. \quad (23.8.14)$$

Let $s = \sum_1^r \alpha_i \chi_{A_i}$, $(A_i)_1^r \subset \mathcal{B}_c(T)$. Then by (b)(i) and (b)(ii) and by Theorem 11.8(v) and Remark 12.5 of [P10] there exists n_2 such that

$$|\alpha_i||\int_{A_i} f_n d(x^* \circ \mathbf{m}) - \int_{A_i} f d(x^* \circ \mathbf{m})| < \frac{\epsilon}{3r} \quad (23.8.15)$$

for $i = 1, 2, ..., r$ and for $n \geq n_2$. Then by (23.8.15) we have

$$|\int_T s f_n d(x^* \circ \mathbf{m}) - \int_T s f d(x^* \circ \mathbf{m})| \leq \sum_{i=1}^r |\alpha_i||\int_{A_i} (f_n - f) d(x^* \circ \mathbf{m})| < \frac{\epsilon}{3} \quad (23.8.16)$$

for $n \geq n_2$. Then by (23.8.14), by Proposition 10.12(iv), by Theorem 11.8(v) and Remark 12.5 of [P10] we have

$$\begin{aligned} &|\int_T sf_n d(x^* \circ \mathbf{m}) - \int_T h_{x^*} f_n d(x^* \circ \mathbf{m})| \\ &= |\int_T (s - h_{x^*}) d(x^* \circ \boldsymbol{\gamma}_n)| \\ &\leq ||s - h_{x^*}||_T v(x^* \circ \boldsymbol{\gamma}_n)(T) \\ &= ||s - h_{x^*}||_T ||\boldsymbol{\gamma}_n||_{q_{x^*}}(T) < \frac{\epsilon}{3} \qquad (22.8.17) \end{aligned}$$

for $n \in \mathbb{N} \cup \{0\}$. Then by (b)(i) and by Theorems 11.8(v) and 11.9(ii) and Remark 12.5 of [P10], and by (23.8.16) and (23.8.17),

$$x^*(\int_T f_n g d\mathbf{m} - \int_T fg d\mathbf{m}) \to 0$$

as $n \to \infty$ for each $x^* \in X^*$ and hence (b)(iii) holds.

Thus Theorem 23.8 generalizes Theorem 23.6 to $\delta(\mathcal{C})$-regular σ-additive quasicomplete lcHs valued vector measures when $f_n \to f$ $\mathbf{m}$-a.e. in T. In order to generalize the said theorem when f_n doesn't satisfy the $\mathbf{m}$-a.e. convergence hypothesis, we restrict $\mathbf{m}$ to be Banach space-valued. To this end, we adapt the proofs of Lemmas 1, 2 and 3 on pp.126-129 of [T] and then we give a stronger vector measure version of Theorem 4.9 of [T] in Theorem 23.12 below.

Lemma 23.9. Let X be a Banach space and let $\mathbf{m} : \delta(\mathcal{C}) \to X$ be σ-additive and $\delta(\mathcal{C})$-regular. Let $(f_n)_1^\infty \subset \mathcal{L}_1(\mathbf{m})$. Then there exists a sequence $(K_n)_1^\infty \subset \mathcal{C}$ such that each f_n vanishes in $T \backslash \bigcup_1^\infty K_n$.

Proof. By Lemma 23.7(i), for each $n \in \mathbb{N}$, there exists $B_n \in \mathcal{B}_c(T)$ such that $N(f_n) \subset B_n$ so that $\bigcup_1^\infty N(f_n) \subset \bigcup_1^\infty B_n \in \mathcal{B}_c(T)$. Since $\bigcup_1^\infty B_n$ is σ-bounded, there exists a sequence $(K_n)_1^\infty \subset \mathcal{C}$ such that $\bigcup_1^\infty B_n \subset \bigcup_1^\infty K_n$. Then $f_n = 0$ in $T \backslash \bigcup_1^\infty K_k$ for each n.

Lemma 23.10. Let X and $\mathbf{m}$ be as in Lemma 23.9 and let H be a norm determining set in X^*. Given a sequence $(K_n)_1^\infty \subset \mathcal{C}$, there exists a sequence $(x_n^*)_1^\infty \subset H$ such that every σ-Borel set $A \subset \bigcup_1^\infty K_n$ is $\mathbf{m}$-null whenever A is $(x_n^* \circ \mathbf{m})$-null for each $n \in \mathbb{N}$.

Proof. By Lemma 18.2 of [P11], $A \cap K_n \in \delta(\mathcal{C})$ for $n \in \mathbb{N}$. Choose a relatively compact open set V_n such that $K_n \subset V_n$. Arguing as in the proof of Lemma 22.2, we can find a sequence $(x_{n,r}^*)_{r=1}^\infty \subset H$ such that $E \in \mathcal{B}(V_n)$ is $\mathbf{m}$-null whenever E is $(x_{n,r}^* \circ \mathbf{m})$-null for all $r \in \mathbb{N}$. Let $(x_n^*)_1^\infty = \{x_{n,r}^* : n, r \in \mathbb{N}\}$. Then A is $\mathbf{m}$-null whenever A is $(x_n^* \circ \mathbf{m})$-null for each $n \in \mathbb{N}$ since $A \cap K_n \in \mathcal{B}(V_n)$ for all n and $A = \bigcup_1^\infty (A \cap K_n)$.

Lemma 23.11. Let $\mu_k : \delta(\mathcal{C}) \to \mathbb{K}$, $k \in \mathbb{N}$, be σ-additive and $\delta(\mathcal{C})$-regular and let $(f_n)_1^\infty \subset \bigcap_{k=1}^\infty \mathcal{L}_1(\mu_k)$. Suppose $\lim_n \int_U f_n d\mu_k \in \mathbb{K}$ for each $k \in \mathbb{N}$ and for each open Baie set U in T. Then there exists a sequence $(g_n)_1^\infty$ such that each g_n is a convex combination of $(f_k)_{k\geq n}$ and such that $(g_n)_1^\infty$ converges in mean in $\mathcal{L}_1(\mu_k)$ and also converges pointwise μ_k-a.e. in T for each $k \in \mathbb{N}$.

Proof. By hypothesis and by Theorem 23.6, for each $k \in \mathbb{N}$, (f_n) converges weakly to some $h_k \in \mathcal{L}_1(\mu_k)$. Let us embed $\bigcap_{k=1}^\infty \mathcal{L}_1(\mu_k)$ in the diagonal of the product space $P = \Pi_{k=1}^\infty \mathcal{L}_1(\mu_k)$. For each f_n, let $\widetilde{f_n} = (f_n, f_n, ...) \in P$. Clearly, P is a pseudo-metrizable locally convex space with the pseudo-norm ρ given by

$$\rho((\varphi_k)_1^\infty, (\psi_k)_1^\infty) = \sum_{k=1}^\infty \frac{1}{2^k} \frac{\int_T |\varphi_k - \psi_k| dv(\mu_k)}{1 + \int_T |\varphi_k - \psi_k| dv(\mu_k)}$$

for $(\varphi_k)_1^\infty, (\psi_k)_1^\infty \in P$. Then by Theorem 4.3, Ch. IV of [Scha], $(\widetilde{f_n})_{n=1}^\infty$ converges weakly to some $\hat{h} = (h_k)_{k=1}^\infty \in P$. Then $(\widetilde{f_n})_{n\geq k}$ converges to $\hat{h}$ weakly for each k and hence by the Hahn-Banach theorem (see the proof of Theorem 3.13 of [Ru2]), $\hat{h} = (h_i)_1^\infty$ belongs to the closed convex hull of $(\widetilde{f_n})_{n\geq k}$ for each k. Then there exists a sequence $(\widetilde{g_n})$ such that $\widetilde{g_n} \in co(\widetilde{f_k}, \widetilde{f_{k+1}}, ...)$ with $\rho(\widetilde{g_n}, \hat{h}) \to 0$. Let $g_n = \sum_{i=n}^{N(n)} \alpha_i f_i$, $\alpha_i \geq 0$, $\sum_{i=n}^{N(n)} \alpha_i = 1$ so that $\widetilde{g_n} = (g_n, g_n, ...)$. Then $g_n \to h_k$ in mean in $\mathcal{L}_1(\mu_k)$ for each k.

For $k = 1$, there exists a subsequence $(g_{1,r})_{r=1}^\infty$ of $(g_n)_1^\infty$ such that $g_{1,r} \to h_1$ μ_1-a.e. in T. Proceeding successively, there exists a subsequence $(g_{n,r})_{r=1}^\infty$ of $(g_{n-1,r})_{r=1}^\infty$ such that $g_{n,r} \to h_n$ μ_n-a.e. in T. Then the diagonal sequence $(g_{n,n})_{n=1}^\infty$ is a subsequence of $(g_n)_{n=1}^\infty$ and converges to h_k μ_k-a.e. in T for each k. Clearly, $(g_{n,n})_{n=1}^\infty$ also converges in mean to h_k in $\mathcal{L}_1(\mu_k)$ for each k. Hence the lemma holds.

Theorem 23.12 (Full generalization of Theorem 23.6 to Banach space-valued measures on $\delta(\mathcal{C})$. Let X be a Banach space with toplogy τ and let H be a norm determining set in X^* with the Orlicz property. Let $\mathbf{m} : \delta(\mathcal{C}) \to X$ be σ-additive and $\delta(\mathcal{C})$-regular.

(a) Suppose $(f_n)_1^\infty \subset \mathcal{L}_1(\mathbf{m})$ is such that for each open Baire set U in T, there exists a vector $x_U \in X$ such that $\int_U f_n d\mathbf{m} \to x_U$ in τ. Then the following hold:

(a)(i) There exists a function $f \in \mathcal{L}_1(\mathbf{m})$ such that $\int_U f_n d\mathbf{m} \to \int_U f d\mathbf{m}$ in τ.

(a)(ii) f in (a)(i) is unique upto $\mathbf{m}$-a.e. in T.

(a)(iii) For $A \in \mathcal{B}(T)$, $\int_A f_n d\mathbf{m} \to \int_A f d\mathbf{m}$ in τ.

(a)(iv) For each bounded $\mathbf{m}$-measurable scalar function g on T, $\int_T f_n g d\mathbf{m} \to \int_T f g d\mathbf{m}$ in τ.

(b) Suppose $(f_n)_1^\infty \subset \mathcal{L}_1(\mathbf{m})$ is such that for each open Baire set U in T, there exists a vector $x_U \in X$ such that $\int_U f_n d\mathbf{m} \to x_U$ in $\sigma(X, H)$. Then the following hold:

(b)(i) There exists $f \in \mathcal{L}_1(\mathbf{m})$ such that $\int_U f_n d\mathbf{m} \to \int_U f d\mathbf{m}$ in $\sigma(X, H)$.

(b)(ii) f in (b)(i) is unique upto $\mathbf{m}$-a.e. in T.

(b)(iii) For $A \in \mathcal{B}(T)$, $\int_A f_n d\mathbf{m} \to \int_A f d\mathbf{m}$ in $\sigma(X, H)$.

(b)(iv) For each bounded $\mathbf{m}$-measurable scalar function g on T, $\int_T f_n g d\mathbf{m} \to \int_T f g d\mathbf{m}$ in $\sigma(X, H)$.

Proof. Let $H_0 = \{x^* : |x^*| \leq 1\}$. By hypothesis (a) (resp. (b)), and by Theorem 3.5(viii) and Remark 4.3 of [P8], $\lim_n \int_U f_n d(x^* \circ \mathbf{m}) = x^*(x_U) \in \mathbb{K}$ for each $x^* \in H_0$ (resp. in H). By Lemma 23.9, there exists $(K_n)_1^\infty \subset \mathcal{C}$ such that each f_n vanishes in $T \backslash \bigcup_1^\infty K_k$. By Lemma 23.10, we associate the sequence $(K_n)_1^\infty$ with a sequence $(x_n^*)_1^\infty \subset H_0$ (resp. $\subset H$) satisfying the property mentioned in the said lemma. Then by Lemma 23.11 there exists a sequence $(g_n)_1^\infty$ such that each g_n is of the form

$$g_n = \sum_{i=n}^{N(n)} \alpha_i^{(n)} f_i, \; \alpha_i^{(n)} \geq 0 \text{ and } \sum_{i=n}^{N(n)} \alpha_i^{(n)} = 1 \qquad (23.12.1)$$

and such that (g_n) converges in mean in $\mathcal{L}_1(x_i^* \circ \mathbf{m})$ and converges $(x_i^* \circ \mathbf{m})$-a.e. in T for each $i \in \mathbb{N}$. Then by Lemma 23.10, $(g_n)_1^\infty$ converges $\mathbf{m}$-a.e. in T. Let f be the $\mathbf{m}$-a.e. pointwise limit of $(g_n)_1^\infty$.

(a)(i) Let U be an open Baire set in T and let $x_U \in X$ be as in the hypothesis (a). Then, given $\epsilon > 0$, there exists n_0 such that

$$\left| \int_U f_n d\mathbf{m} - x_U \right| < \epsilon \qquad (23.12.2)$$

for $n \geq n_0$. Let $(g_n)_1^\infty$ be as in (23.12.1). Then

$$\left| \int_U g_n d\mathbf{m} - x_U \right| = \left| \sum_{i=n}^{N(n)} (\alpha_i^{(n)} \int_U f_i d\mathbf{m} - \alpha_i^{(n)} x_U) \right| \leq \sum_{i=n}^{N(n)} \alpha_i^{(n)} \epsilon = \epsilon$$

for $n \geq n_0$ and hence

$$\lim_n \int_U g_n d\mathbf{m} = x_U \text{ in } \tau. \qquad (23.12.3)$$

Then by Theorem 23.8(a), $f \in \mathcal{L}_1(\mathbf{m})$ and

$$\int_U g_n d\mathbf{m} \to \int_U f d\mathbf{m} \text{ in } \tau. \qquad (23.12.4)$$

Then by (23.12.3) and (23.12.4) we have

$$\int_U f d\mathbf{m} = x_U$$

and hence by hypothesis,

$$\lim_n \int_U f_n d\mathbf{m} = \int_U f d\mathbf{m} \text{ in } \tau.$$

Thus (a)(i) holds.

(b)(i) Let $x^* \in H$. By hypothesis (b) and by Theorem 3.5(v) and Remark 4.3 of [P8],

$$\lim_n \int_U f_n d(x^* \circ \mathbf{m}) = \lim_n x^*(\int_U f_n d\mathbf{m}) = x^*(x_U) \qquad (23.12.5)$$

for each open Baire set U in T. Hence, given $\epsilon > 0$, there exists n_1 such that

$$|\int_U f_n d(x^* \circ \mathbf{m}) - x^*(x_U)| < \epsilon \qquad (23.12.6)$$

for $n \geq n_1$. Then by (23.12.1) and (23.12.6) we have

$$|\int_U g_n d(x^* \circ \mathbf{m}) - x^*(x_U)| = |\sum_{i=n}^{N(n)} (\alpha_i^{(n)} \int_U f_i d(x^* \circ \mathbf{m}) - \alpha_i^{(n)} x^*(x_U))| < \epsilon$$

for $n \geq n_1$. Hence

$$\lim_n \int_U g_n d(x^* \circ \mathbf{m}) = x^*(x_U) \qquad (23.12.7)$$

for $x^* \in H$. By hypothesis and by Theorem 4.4(i) of [P2] there exists $\theta \in \mathcal{K}(T)^*$ such that $\mu_\theta = x^* \circ \mathbf{m}$ and hence by (23.12.7) and by Theorem 23.6, $f \in \mathcal{L}_1(x^* \circ \mathbf{m})$ and

$$\lim_n \int_U g_n d(x^* \circ \mathbf{m}) = \int_U f d(x^* \circ \mathbf{m}) \qquad (23.12.8)$$

for $x^* \in H$. Then by (23.12.7) and (23.12.8)

$$\int_U f d(x^* \circ \mathbf{m}) = x^*(x_U)$$

for $x^* \in H$. Consequently, by Theorem 22.4, $f \in \mathcal{L}_1(\mathbf{m})$ and by by Theorem 3.5(v) and Remark 4.3 of [P8],

$$x^*(\int_U f d\mathbf{m}) = \int_U f d(x^* \circ \mathbf{m}) = x^*(x_U)$$

for $x^* \in H$. As H is norm determining, we conclude that

$$\int_U f d\mathbf{m} = x_U \qquad (23.12.9)$$

and hence by (23.12.5) and (23.12.9), (b)(i) holds.

(a)(ii) (resp. (b)(ii))

Claim 1. Let μ_1 and μ_2 be in $M(T)$. If $\mu_1(U) = \mu_2(U)$ for each open Baire set U in T, then $\mu_1 = \mu_2$.

In fact, Let $\nu_i = \mu_i|_{\mathcal{B}_0(T)}$ for $i = 1, 2$. By Proposition 1 of [P13], ν_1 and ν_2 are Baire regular. For $E \in \mathcal{B}_0(T)$, let $\mathcal{U}_E = \{U \in \mathcal{B}_0(T) : U \text{ open}, U \supset E\}$ and let $U_1 \geq U_2$ for $U_1, U_2 \in \mathcal{U}_E$ if $U_1 \subset U_2$. Then $\mathcal{U}_E$ is a directed set and by the Baire regularity of ν_1 and ν_2 we have

$$\nu_1(E) = \lim_{U \in \mathcal{U}_E, U \to E} \nu_1(U) = \lim_{U \in \mathcal{U}_E, U \to E} \nu_2(U) = \nu_2(E)$$

and hence $\nu_1 = \nu_2$ on $\mathcal{B}_0(T)$. Then the claim is immediate from the uniqueness part of Proposition 1 of [P13].

Suppose h is another function in $\mathcal{L}_1(\mathbf{m})$ such that $\lim_n \int_U f_n d\mathbf{m} = \int_U h d\mathbf{m}$ in τ (resp. in $\sigma(X, H)$) for open Baire sets U in T. Let $x^* \in H_0$ (resp. $x^* \in H$). By hypothesis and by Theorem 4.4(i) of [P2] there exists $\theta \in \mathcal{K}(T)^*$ such that $x^* \circ \mathbf{m} = \mu_\theta$ and as seen in the proof of Theorem 23.6, $\int_{(\cdot)} h d(x^* \circ \mathbf{m})$ and $\int_{(\cdot)} f d(x^* \circ \mathbf{m})$ belong to $M(T)$. Then by hypothesis (a) (resp. (b)), by Theorem 3.5(v) and Remark 4.3 of [P8] and by Claim 1 above, ν is a null measure where $\nu(A) = \int_A (h - f) d(x^* \circ \mathbf{m})$ for $A \in \mathcal{B}(T)$. Then by Proposition 2.11 of [P8] and by Theorem 1.39 of [Ru1], $h = f$ $(x^* \circ \mathbf{m})$-a.e. in T. This holds for each $x^* \in H_0$ (resp. $x^* \in H$) and hence by Lemmas 23.9 and 23.10, $h = f$ $\mathbf{m}$-a.e. in T. Therefore, (a)(ii) (resp. (b)(ii)) holds.

(b)(iii) and (b)(iv) Let $f_0 = f$. For $x^* \in X^*$, by (b)(i) and by Theorem 3.5(v) and Remark 4.3 of [P8] we have

$$\lim_n \int_U f_n d(x^* \circ \mathbf{m}) = \int_U f d(x^* \circ \mathbf{m})$$

for each open Baire set U in T. Moreover, by hypothesis and by Theorem 4.4(i) of [P2] there exists $\theta \in \mathcal{K}(T)^*$ such that $x^* \circ \mathbf{m} = \mu_\theta$ and hence by Theorem 23.6 and by the uniqueness of f in $L_1(x^* \circ \mathbf{m})$, we have

$$\lim_n \int_T g f_n d(x^* \circ \mathbf{m}) = \int_T g f d(x^* \circ \mathbf{m}) \qquad (23.12.9)$$

for each bounded $\mathbf{m}$-measurable scalar function g on T. Then by (23.12.9), by Theorem 3.5 and Remark 4.3 of [P8] and by the fact that $(f_n)_{n=0}^\infty \subset \mathcal{L}_1(\mathbf{m})$, we conclude that $\int_T f_n g d\mathbf{m} \to \int_T f g d\mathbf{m}$ in $\sigma(X, H)$. Hence (b)(iv) holds. Let $A \in \mathcal{B}(T)$ and let B_n, $n \in \mathbb{N} \cup \{0\}$, be as in the proof of Theorem 23.8(a). Let $B = \bigcup_{n=0}^\infty B_n$. Then, by (b)(iv) we have $x^*(\int_A f_n d\mathbf{m}) = x^*(\int_{A \cap B} f_n d\mathbf{m}) \to x^*(\int_{A \cap B} f d\mathbf{m}) = x^*(\int_A f d\mathbf{m})$ for $x^* \in H$. Hence (b)(iii) also holds.

(a)(iii) Let B_n, $n \in \mathbb{N} \cup \{0\}$, B and f_0 be as in the proof of (b)(iii) and (b)(iv). Let $\boldsymbol{\gamma}_n, n \in \mathbb{N} \cup \{0\}$, be as in the proof of Theorem 23.8(a)(ii). Then by hypothesis, $\lim_n \boldsymbol{\gamma}_n(U) \in X$ in τ for each open Baire set U in T and hence, by Theorem 18.21 of [P11] there exists a unique X-valued σ-additive measure $\boldsymbol{\gamma}$ on $\mathcal{B}(T)$ such that

$$\lim_n \int_T g d\boldsymbol{\gamma}_n = \int_T g d\boldsymbol{\gamma} \qquad (23.12.10)$$

in τ for each bounded Borel measurable scalar function g on T. Then by Claim 1 in the proof of Theorem 23.8, by Theorem 3.5 and Remark 4.3 of [P8] and (23.12.10) we have

$$\lim_n \int_T gf_n d(x^* \circ \mathbf{m}) = \lim_n x^*(\int_T gf_n d\mathbf{m}) = \lim_n x^*(\int_T gd\boldsymbol{\gamma}_n) = x^*(\int_T gd\boldsymbol{\gamma})$$

for $x^* \in X^*$. By (23.12.9) which also holds for $x^* \in X^*$, and by Theorem 3.5 and Remark 4.3 of [P8] we have

$$\lim_n \int_T gf_n d(x^* \circ \mathbf{m}) = \int_T gfd(x^* \circ \mathbf{m}) = x^*(\int_T gfd\boldsymbol{\gamma}_0)$$

and hence

$$x^*(\int_T gd\boldsymbol{\gamma}) = x^*(\int_T gd\boldsymbol{\gamma}_0)$$

for $x^* \in X^*$. Then by the Hahn-Banach theorem and by Claim 1 in the proof of Theorem 23.8 we have

$$\int_T gd\boldsymbol{\gamma} = \int_T gd\boldsymbol{\gamma}_0 = \int_T gfd\mathbf{m} \qquad (23.12.11)$$

for each bounded Borel measurable function g on T and hence by (23.12.10) and (23.12.11) we have

$$\lim_n \int_T f_n gd\mathbf{m} = \int_T fgd\mathbf{m} \quad \text{in } \tau. \qquad (23.12.12)$$

If g is a bounded $\mathbf{m}$-measurable scalar function on T, then there exists a bounded Borel measurable function h such that $g = h$ $\mathbf{m}$-a.e. in T and hence by (23.12.12), (a)(iv) holds. Moeover, (a)(iii) is immediate from (a)(iv).

Remark 23.13. Suppose X is a quasicomplete lcHs with topology τ and $\mathbf{m} : \mathcal{B}(T) \to X$ is σ-additive and Borel regular. Then, in the light of theorem 22.11, the analogue of Theorem 23.8 for $\mathbf{m}$ holds here verbatim. Moreover, if X is a Banach space, then in view of Theorem 21.1(iv) applied to $N(f) \in \widetilde{\mathcal{B}(T)}$, Lemma 23.9 holds here with f_n vanishing in $\mathbf{m}$-a.e. in $T \setminus \bigcup_1^\infty K_k$. Then the analogue of Theorem 23.12 for $\mathbf{m}$ holds here verbatim if we use Theorem 22.10 in place of Theorem 22.4. The details are left to the reader.

24. DUALS OF $\mathcal{L}_1(\mathbf{m})$ AND $\mathcal{L}_1(\mathbf{n})$

Let X be a Banach space. Suppose $\mathbf{m} : \mathcal{B}(T) \to X$ (resp. $\mathbf{n} : \delta(\mathcal{C}) \to X$) is σ-additive and $\mathcal{B}(T)$-regular (resp. and $\delta(\mathcal{C})$-regular). The present section is devoted to the study of the duals of $\mathcal{L}_1(\mathbf{m})$ and $\mathcal{L}_1(\mathbf{n})$. Also are given vector measure analogues of Theorem 4.1 and Proposition 5.9 of [T].

Lemma 24.1. Let X be a Banach space and let $\mathbf{m} : \mathcal{B}(T) \to X$ (resp. $\mathbf{n} : \delta(\mathcal{C}) \to X$) be σ-additive and $\mathcal{B}(T)$-regular (resp. and $\delta(\mathcal{C})$-regular). If $u \in \mathcal{L}_1(\mathbf{m})^*$ (resp. $v \in \mathcal{L}_1(\mathbf{n})^*$), then there exists a unique σ-additive and $\mathcal{B}(T)$-regular (resp. and $\delta(\mathcal{C})$-regular) scalar measure η_u on $\mathcal{B}(T)$ (resp. ζ_v on $\delta(\mathcal{C})$) such that

$$u(f) = \int_T f d\eta_u \text{ and } \int_T |f| dv(\eta_u, \mathcal{B}(T)) \leq ||u|| \mathbf{m}_1^\bullet(f, T) \qquad (24.1.1)$$

for $f \in \mathcal{L}_1(\mathbf{m})$ where

$$||u|| = \sup\{|u(f)| : f \in \mathcal{L}_1(\mathbf{m}), \mathbf{m}_1^\bullet(f, T) \leq 1\}$$

(

$$\text{resp.}\, v(f) = \int_T f d\zeta_v \text{ and } \int_T |f| dv(\zeta_v, \delta(\mathcal{C})) \leq ||v|| \mathbf{n}_1^\bullet(f, T) \qquad (24.1.1')$$

for $f \in \mathcal{L}_1(\mathbf{n})$ where

$$||v|| = \sup\{|v(f)| : f \in \mathcal{L}_1(\mathbf{n}), \mathbf{n}_1^\bullet(f, T) \leq 1\}).$$

Proof. Let $u \in \mathcal{L}_1(\mathbf{m})^*$ (resp. $v \in \mathcal{L}_1(\mathbf{n})^*$). Then

$$|u(f)| \leq ||u|| \mathbf{m}_1^\bullet(f, T) \qquad (24.1.2)$$

for $f \in \mathcal{L}_1(\mathbf{m})$

$$(\text{resp.}\, |v(f)| \leq ||v|| \mathbf{n}_1^\bullet(f, T) \qquad (24.1.2')$$

for $f \in \mathcal{L}_1(\mathbf{n})$).

Let $\varphi \in \mathcal{K}(T)$ (resp. $\varphi \in C_c(T, C)$ with $C \in \mathcal{C}$- see Notation 19.1 of [P11]). Then, as $\varphi \in \mathcal{L}_1(\mathbf{m})$ (resp. $\mathcal{L}_1(\mathbf{n})$) by Lemma 20.8, by (24.1.2) (resp. by (24.1.2')), we have

$$\begin{aligned} |u(\varphi)| \leq ||u|| \mathbf{m}_1^\bullet(\varphi, T) &= ||u|| (\sup_{|x^*| \leq 1} \int_T |\varphi| dv(x^* \circ \mathbf{m})) \\ &\leq ||u|| ||\varphi||_T ||\mathbf{m}||(T) \end{aligned}$$

$$(\text{resp.}\, |v(\varphi)| \leq ||v|| \mathbf{n}_1^\bullet(\varphi, T) \leq ||v|| ||\varphi||_T ||\mathbf{n}||(C)).$$

Hence $u|_{\mathcal{K}(T)} \in \mathcal{K}(T)_b^*$ (see pp. 65 and 69 of [P2]). Let $u|_{\mathcal{K}(T)} = \theta_u$ (by Theorem 4.4(i) oby f [P2], this is determined uniquely by u)and let $\eta_u = \mu_{\theta_u}|_{\mathcal{B}(T)}$, where μ_{θ_u} is the complex Radon measure induced by θ_u in the sense of Definition 4.3 of [P1]. Then η_u is σ-additive on $\mathcal{B}(T)$ and is Borel regular by Theorem 5.3 of [P2] and moreover,

$$u(\varphi) = \int_T \varphi d\eta_u = \int_T \varphi d\mu_{\theta_u} = \theta_u(\varphi) \qquad (24.1.3)$$

for $\varphi \in C_0(T)$. (Resp. $v|_{\mathcal{K}(T)} \in \mathcal{K}(T)^*$. Let $v|_{\mathcal{K}(T)} = \theta'_v$. By Theorem 4.4(i) of [P2], this is uniquely determined bt v and $\zeta_v = \mu_{\theta'_v}|_{\delta(\mathcal{C})}$ is σ-additive on $\delta(\mathcal{C})$ and is $\delta(\mathcal{C})$-regular by Theorem 4.7 of [P1] and

$$v(\varphi) = \int_T \varphi d\zeta_v = \int_T \varphi d\mu_{\theta'_v} = \theta'_v(\varphi) \qquad (24.1.3')$$

for $\varphi \in \mathcal{K}(T)$).

Then by (12) on p. 55 of [B] and by (24.1.2) and (24.1.3) (resp. and by (24.1.2') and (24.1.3')) we have

$$\begin{aligned} |\theta_u|(|\varphi|) &= \sup_{\Psi\in\mathcal{K}(T),|\Psi|\leq|\varphi|} |\theta_u(\Psi)| \\ &= \sup_{\Psi\in\mathcal{K}(T),|\Psi|\leq|\varphi|} |u(\Psi)| \\ &\leq ||u|| \sup_{\Psi\in\mathcal{K}(T),|\Psi|\leq|\varphi|} \mathbf{m}_1^\bullet(\Psi,T) \\ &= ||u||\mathbf{m}_1^\bullet(\varphi,T) \quad (24.1.4) \end{aligned}$$

(resp.

$$\begin{aligned} |\theta'_v|(|\varphi|) = \sup_{\Psi\in\mathcal{K}(T),|\Psi|\leq|\varphi|} |v(\Psi)| &\leq ||v|| \sup_{\Psi\in\mathcal{K}(T),|\Psi|\leq|\varphi|} \mathbf{n}_1^\bullet(\Psi,T) \\ &= ||v||\mathbf{n}_1^\bullet(\varphi,T) \qquad (24.1.4')) \end{aligned}$$

for $\varphi \in \mathcal{K}(T)$. Then by Theorems 4.7 and 4.11 of [P1] and by the last part of Theorem 3.3 of [P2] (resp. by Theorems 4.7 and 4.11 of [P1]) and by (24.1.4) (resp. by (24.1.4')) we have

$$|\theta_u|(|\varphi|) = \int_T |\varphi| d\mu_{|\theta_u|} = \int_T |\varphi| dv(\mu_\theta, \mathcal{B}(T)) \leq ||u||\mathbf{m}_1^\bullet(\varphi,T) \quad (24.1.5)$$

$$(\text{resp. } |\theta'_v|(|\varphi|) = \int_T |\varphi| d\mu_{|\theta'_v|} = \int_T |\varphi| dv(\mu_{\theta'_v}, \delta(\mathcal{C})) \leq ||v||\mathbf{n}_1^\bullet(\varphi,T) \quad (24.1.5'))$$

for $\varphi \in \mathcal{K}(T)$.

Let $\Im^+$ be the set of all non negative lower semicontinuous functions on T.

Claim 1. For $f \in \Im^+ \cap \mathcal{L}_1(\mathbf{m})$ (resp. $f \in \Im^+ \cap \mathcal{L}_1(\mathbf{n})$),

$$|\theta_u|^*(f) \leq ||u||\mathbf{m}_1^\bullet(f,T) \qquad (24.1.6)$$

$$(\text{resp. } |\theta'_v|^*(f) \leq ||v||\mathbf{n}_1^\bullet(f,T) \qquad (24.1.6'))$$

where $|\theta_u|^*$ and $|\theta'_v|^*$ are as in Definition 1, § 1, Ch. IV of [B].

In fact, by the said definition of [B] and by (24.1.5) we have

$$|\theta_u|^*(f) = \sup_{\varphi\in\mathcal{K}(T)^+,\varphi\leq f} |\theta_u|(\varphi) \leq ||u|| \sup_{\varphi\in\mathcal{K}(T)^+,\varphi\leq f} \mathbf{m}_1^\bullet(\varphi,T) \leq ||u||\mathbf{m}_1^\bullet(f,T)$$

for $f \in \Im^+ \cap \mathcal{L}_1(\mathbf{m})$. Similarly, by (24.1.5'), (24.1.6') holds.

Claim 2. Let $f \in \mathcal{L}_1(\mathbf{m})$ (resp. $f \in \mathcal{L}_1(\mathbf{n})$). Then

$$\int_T |f| dv(\eta_u, \mathcal{B}(T)) = \int_T |f| dv(\mu_{\theta_u}, \mathcal{B}(T)) \leq ||u|| \mathbf{m}_1^\bullet(f, T) \quad (24.1.7)$$

$$\text{(resp.} \int_T |f| dv(\zeta_v, \delta(\mathcal{C})) = \int_T |f| dv(\mu_{\theta'_v}, \delta(\mathcal{C})) \leq ||v|| \mathbf{n}_1^\bullet(f, T). \quad (24.1.7')).$$

In fact, as $\eta_u = \mu_{\theta_u}|_{\mathcal{B}(T)}$ (resp. $\zeta_v = \mu_{\theta'_v}|_{\delta(\mathcal{C})}$) it suffices to prove the claim for μ_{θ_u} (resp. $\mu_{\theta'_v}$). Note that $|f| \in \mathcal{L}_1(\mathbf{m})$ (resp. $\mathcal{L}_1(\mathbf{n})$) by Theorem 3.5(vii) and Remark 4.3 of [P8] . Given $\epsilon > 0$, by Theorem 17.2 (resp. by Theorem 17.3) of [P11], there exist functions g and h on T such that $0 \leq g \leq |f| \leq h$ $\mathbf{m}$-a.e. (resp. $\mathbf{n}$-a.e.) in T, g is upper semicontinuous, bounded and $\mathbf{m}$-integrable in T (resp. and $\mathbf{n}$-integrable in T), h is lower semicontinuous and $\mathbf{m}$-integrable in T (resp. and $\mathbf{n}$-integrable in T) and

$$\mathbf{m}_1^\bullet(h - g, T) < \frac{\epsilon}{||u||} \qquad (24.1.8)$$

(resp. g and h are $\mathcal{B}_c(T)$-measurable and

$$\mathbf{n}_1^\bullet(h - g, T) < \frac{\epsilon}{||v||} \qquad (24.1.8')).$$

As h and $-g$ are lower semicontinuous, $h - g \in \Im^+ \cap \mathcal{L}_1(\mathbf{m})$ (resp. $h - g \in \Im^+ \cap \mathcal{L}_1(\mathbf{n})$) by Theorems 3.3 and 3.4, § 3, Ch. III of [MB]. Then by Proposition 1, no.2, § 4, Ch. IV of [B] and by (24.1.6) and (24.1.8) (resp. and by (24.1.6') and (24.1.8')) we have

$$0 \leq \int_T (h - g) dv(\mu_{\theta_u}, \mathcal{B}(T)) = |\theta_u|^*(h - g) \leq ||u|| \mathbf{m}_1^\bullet(h - g, T) < \epsilon \qquad (24.1.9)$$

$$\text{(resp.} \, 0 \leq \int_T (h - g) dv(\mu_{\theta'_v}, \delta(\mathcal{C})) = |\theta'_v|^*(h - g) \leq ||v|| \mathbf{n}_1^\bullet(h - g, T) < \epsilon \quad (24.1.9')).$$

Then, as $h \in \Im^+ \cap \mathcal{L}_1(\mathbf{m})$ (resp. $h \in \Im^+ \cap \mathcal{L}_1(\mathbf{n})$), by Claim 1, by Proposition 1, no. 2, § 4, Ch. IV of [B] and by (24.1.9) (resp. and by (24.1.9')) we have

$$\begin{aligned}
\int_T |f| dv(\mu_\rho, \mathcal{R}) &\leq \int_T h dv(\mu_\rho, \mathcal{R}) \\
&= |\rho|^*(h) \leq ||w|| \boldsymbol{\omega}_1^\bullet(h, T) \\
&\leq ||w|| \{\boldsymbol{\omega}_1^\bullet(h - g, T) + \boldsymbol{\omega}_1^\bullet(g, T)\} \\
&< \epsilon + ||w|| \boldsymbol{\omega}_1^\bullet(g, T) \\
&\leq \epsilon + ||w|| \boldsymbol{\omega}_1^\bullet(f, T)
\end{aligned}$$

since $0 \leq g \leq |f|$, where $\mathcal{R} = \mathcal{B}(T)$, $w = u$ and $\rho = \theta_u$ (resp. $\mathcal{R} = \delta(\mathcal{C})$, $w = v$ and $\rho = \theta'_v$). As $\epsilon > 0$ is arbitrary, the claim holds.

Claim 3. For $f \in \mathcal{L}_1(\mathbf{m})$, $u(f) = \int_T f d\eta_u = \int_T f d\mu_{\theta_u}$ and for $f \in \mathcal{L}_1(\mathbf{n})$, $v(f) = \int_T f d\zeta_v = \int_T f d\mu_{\theta'_v}$.

In fact, it suffices to prove the claim for $f \in \mathcal{L}_1(\mathbf{m})$ since the proof for $f \in \mathcal{L}_1(\mathbf{n})$ is similar.

Let $f \in \mathcal{L}_1(\mathbf{m})$. Since $(C_c(T), \mathbf{m}_1^\bullet(\cdot, T))$ is dense in $\mathcal{L}_1(\mathbf{m})$ by Theorem 20.10, there exists $(\varphi_n)_1^\infty \subset C_c(T)$ such that

$$\lim_n \mathbf{m}_1^\bullet(f - \varphi_n, T) = 0. \qquad (24.1.10)$$

Then by Claim 2 and by (24.1.10), we have

$$\lim_n \int_T |\varphi_n - f| dv(\mu_{\theta_u}, \mathcal{B}(T)) \leq ||u||(\lim_n \mathbf{m}_1^\bullet(f - \varphi_n, T)) = 0$$

and hence by (24.1.3) we have

$$\int_T f d\mu_{\theta_u} = \lim_n \int_T \varphi_n d\mu_{\theta_u} = \lim_n \int_T \varphi_n d\eta_u = \lim_n u(\varphi_n) = u(f)$$

since $u \in \mathcal{L}_1(\mathbf{m})^*$. Hence the claim holds.
Now the lemma is immediate from Claims 2 and 3.

Theorem 24.2. Let X be a Banach space and let $\mathbf{m} : \mathcal{B}(T) \to X$ be σ-additive and $\mathcal{B}(T)$-regular (resp. let $\mathbf{n} : \delta(\mathcal{C}) \to X$ be σ-additive and $\delta(\mathcal{C})$-regular). Let $Y = \{\eta : \mathcal{B}(T) \to \boldsymbol{K}$: σ-additive and $\mathcal{B}(T)$-regular such that there exists a constant M satisfying $\int_T |f| dv(\eta, \mathcal{B}(T)) \leq M\mathbf{m}_1^\bullet(f, T)$ for $f \in \mathcal{L}_1(\mathbf{m})\}$ and let $Z = \{\zeta : \delta(\mathcal{C}) \to \boldsymbol{K}$: σ-additive and $\delta(\mathcal{C})$-regular such that there exists a constant M satisfying $\int_T |f| dv(\zeta, \delta(\mathcal{C})) \leq M\mathbf{n}_1^\bullet(f, T)$ for $f \in \mathcal{L}_1(\mathbf{n})\}$. Let $|||\eta||| = \sup\{|\int_T f d\eta| : f \in \mathcal{L}_1(\mathbf{m}), \mathbf{m}_1^\bullet(f, T) \leq 1\}$ for $\eta \in Y$ and let $|||\zeta||| = \sup\{|\int_T f d\zeta| : f \in \mathcal{L}_1(\mathbf{n}), \mathbf{n}_1^\bullet(f, T) \leq 1\}$ for $\zeta \in Z$. Then:

(i) $\mathcal{L}_1(\mathbf{m})^*$ (resp. $\mathcal{L}_1(\mathbf{n})^*$) is isometrically isomorphic with $(Y, |||\cdot|||)$ (resp. $(Z, |||\cdot|||)$) so that $\mathcal{L}_1(\mathbf{m})^* = Y$ (resp. $\mathcal{L}_1(\mathbf{n})^* = Z$). Consequently, $(Y, |||\cdot|||)$ (resp. $(Z, |||\cdot|||)$) is a Banach space.

(ii) The closed unit ball B_Y of Y (resp. B_Z of Z) is given by $A = \{\eta \in Y : \int_T |f| dv(\eta, \mathcal{B}(T)) \leq \mathbf{m}_1^\bullet(f, T)$ for $f \in \mathcal{L}_1(\mathbf{m})\}$ (resp. $B = \{\zeta \in Z : \int_T |f| dv(\zeta, \delta(\mathcal{C})) \leq \mathbf{n}_1^\bullet(f, T)$ for $f \in \mathcal{L}_1(\mathbf{n})\}$).

(iii) If $B_Y^+ = \{\eta \in B_Y : \eta \geq 0\}$ (resp. $B_Z^+ = \{\zeta \in B_Z : \zeta \geq 0\}$), then

$$\mathbf{m}_1^\bullet(f, T) = \sup_{\eta \in B_Y^+} \int_T |f| d\eta \quad \text{for } f \in \mathcal{L}_1(\mathbf{m})$$

$$(\text{resp. } \mathbf{n}_1^\bullet(f, T) = \sup_{\zeta \in B_Z^+} \int_T |f| d\zeta \quad \text{for } f \in \mathcal{L}_1(\mathbf{n})).$$

Proof. Let $\eta \in Y$ and let $u_\eta : \mathcal{L}_1(\mathbf{m}) \to \boldsymbol{K}$ be given by

$$u_\eta(f) = \int_T f d\eta \qquad (24.2.1)$$

(resp. let $\zeta \in Z$ and let $v_\zeta : \mathcal{L}_1(\mathbf{n}) \to \boldsymbol{K}$ be given by

$$v_\zeta(f) = \int_T f d\zeta \qquad (24.2.1')).$$

Then by hypothesis, there exists $M > 0$ such that

$$|u_\eta(f)| \le \int_T |f| dv(\eta, \mathcal{B}(T)) \le M \mathbf{m}_1^\bullet(f, T) \qquad (24.2.2)$$

for $f \in \mathcal{L}_1(\mathbf{m})$

$$(\text{resp.}\, |v_\zeta(f)| \le \int_T |f| dv(\zeta, \delta(\mathcal{C})) \le M \mathbf{n}_1^\bullet(f, T) \qquad (24.2.2')$$

for $f \in \mathcal{L}_1(\mathbf{n})$). Hence $u_\eta \in \mathcal{L}_1(\mathbf{m})^*$ (resp. $v_\zeta \in \mathcal{L}_1(\mathbf{n})^*$).

Conversely, let $u \in \mathcal{L}_1(\mathbf{m})^*$ (resp. $v \in \mathcal{L}_1(\mathbf{n})^*$). Then by Lemma 24.1, there exists a unique $\eta_u \in Y$ (resp. $\zeta_v \in Z$) such that

$$u(f) = \int_T f d\eta_u \,\text{with} \int_T |f| dv(\eta_u, \mathcal{B}(T)) \le ||u|| \mathbf{m}_1^\bullet(f, T) \,\text{for}\, f \in \mathcal{L}_1(\mathbf{m})$$

$$(\text{resp.}\, v(f) = \int_T f d\zeta_v \,\text{with} \int_T |f| dv(\zeta_v, \delta(\mathcal{C}) \le ||v|| \mathbf{n}_1^\bullet(f, T) \,\text{for}\, f \in \mathcal{L}_1(\mathbf{n})).$$

Let $\Phi : \mathcal{L}_1(\mathbf{m})^* \to Y$ (resp. $\Psi : \mathcal{L}_1(\mathbf{n})^* \to Z$) be given by $\Phi(u) = \eta_u$ (resp. $\Psi(v) = \zeta_v$) so that

$$u(f) = \int_T f d\eta_u, \; f \in \mathcal{L}_1(\mathbf{m}) \qquad (24.1.11)$$

$$(\text{resp.} v(f) = \int_T f d\zeta_v, \; f \in \mathcal{L}_1(\mathbf{n}) \qquad (24.1.11')).$$

Then by Lemma 24.1, Φ (resp. Ψ) is linear since $\alpha\eta_u + \beta\eta_v = \eta_{\alpha u + \beta v}$ on $C_c(T)$ by the Riesz representation theorem and since $C_c(T)$ is dense in $\mathcal{L}_1(\mathbf{m})$ (resp. in $\mathcal{L}_1(\mathbf{n})$) by theorem 20.10. Clearly, Φ and Ψ are injective. To show that Φ (resp. Ψ) is surjective, let $\eta \in Y$ (resp. $\zeta \in Z$). Then u_η (resp. v_ζ) given by $u_\eta(f) = \int_T f d\eta$ for $f \in \mathcal{L}_1(\mathbf{m})$ (resp. $v_\zeta(f) = \int_T f d\zeta$ for $f \in \mathcal{L}_1(\mathbf{n})$) belongs to $\mathcal{L}_1(\mathbf{m})^*$ (resp. $\mathcal{L}_1(\mathbf{n})^*$) and arguing as in the beginning of the proof of Lemma 24.1 one can show that $u_\eta|_{\mathcal{K}(T)} = \theta_{u_\eta} \in \mathcal{K}(T)_b^*$ (resp. $v_\zeta|_{\mathcal{K}(T)} = {\theta'}_{v_\zeta} \in \mathcal{K}(T)^*$) and hence $\int_T \varphi d\eta = u_\eta(\varphi) = \int_T \varphi d\mu_{\theta_{u_\eta}}$ for $\varphi \in C_0(T)$ (resp. $\int_T \varphi d\zeta = v_\zeta(\varphi) = \int_T \varphi d\mu_{{\theta'}_{v_\zeta}}$ for $\varphi \in \mathcal{K}(T)$). Then by the uniqueness part of the Riesz representation theorem for $C_0(T)$ (resp. for $B(V)$ for $V \in \mathcal{V}$ since $\delta(\mathcal{C}) = \bigcup_{V \in \mathcal{V}} \mathcal{B}(V)$) we conclude that $\eta = \mu_{\theta_{u_\eta}}$ (resp. $\zeta = \mu_{{\theta'}_{v_\zeta}}$) so that $u_\eta(\varphi) = \int_T \varphi d\eta = \int_T \varphi d\mu_{\theta_{u_\eta}}$ for $\varphi \in C_c(T)$ (resp. $v_\zeta(\varphi) = \int_T \varphi d\zeta = \int_T \varphi d\mu_{{\theta'}_{v_\zeta}}$ for $\varphi \in C_c(T)$). Since $(C_c(T), \mathbf{m}_1^\bullet(\cdot, T))$ (resp. $C_c(T), \mathbf{n}_1^\bullet(\cdot, T)$) is dense in $\mathcal{L}_1(\mathbf{m})$ (resp. $\mathcal{L}_1(\mathbf{n})$) by

Theorem 20.10, we conclude that $\int_T f d\eta = \int_T f d\mu_{\theta_{u_\eta}}$ for $f \in \mathcal{L}_1(\mathbf{m})$(resp. $\int_T f d\zeta = \int_T f d\mu_{\theta' v_\zeta}$ for $f \in \mathcal{L}_1(\mathbf{n})$) so that $\eta = \Phi(\theta_{u_\eta})$ (resp. $\zeta = \Psi(\theta'_{v_\zeta})$), where we identify θ_{u_η} with an element of $\mathcal{L}_1(\mathbf{m})^*$ (resp. $\mathcal{L}_1(\mathbf{n})^*$) given by $\theta_{u_\eta}(f) = \int_T f d\mu_{\theta_{u_\eta}}$ for $f \in \mathcal{L}_1(\mathbf{m})$ (resp. $\theta'_{v_\zeta}(f) = \int_T f d\mu_{\theta' v_\zeta}$ for $\in \mathcal{L}_1(\mathbf{n})$). Hence Φ (resp. Ψ) is bijective. The remaining parts of (i) are immediate.

(ii) If $\eta \in A$, then

$$|||\eta||| = \sup_{f\in\mathcal{L}_1(\mathbf{m}),\mathbf{m}_1^\bullet(f,T)\le 1} |\int_T f d\eta| \le \sup_{f\in\mathcal{L}_1(\mathbf{m}),\mathbf{m}_1^\bullet(f,T)\le 1} \int_T |f| dv(\eta, \mathcal{B}(T)) \le 1$$

by the definition of A and hence $A \subset B_Y$. Similarly, $|||\zeta||| \le 1$ for $\zeta \in B$ and hence $B \subset B_Z$.

Conversely, let $\eta \in B_Y$. Then $u_\eta(f) = \int_T f d\eta$ for $f \in \mathcal{L}_1(\mathbf{m})$ and

$$||u_\eta|| = \sup_{f\in\mathcal{L}_1(\mathbf{m}),\mathbf{m}_1^\bullet(f,T)\le 1} |\int_T f d\eta| \le 1.$$

Then by (24.1.1), $\int_T |f| dv(\eta, \mathcal{B}(T)) \le ||u_\eta|| \mathbf{m}_1^\bullet(f,T) \le \mathbf{m}_1^\bullet(f,T)$ for $f \in \mathcal{L}_1(\mathbf{m})$ and hence $\eta \in A$. Therefore, $A = B_Y$. Similary, $B = B_Z$.

(iii) By (i), $\boldsymbol{\omega}_1^\bullet(f,T) = \sup_{\eta\in D} |\int_T f d\eta| \le \sup_{\eta\in D} \int_T |f| dv(\eta, \mathcal{R})) \le \sup_{\eta\in D^+} \int_T |f| d\eta \le \boldsymbol{\omega}_1^\bullet(f,T)$ where $\boldsymbol{\omega} = \mathbf{m}$, $f \in \mathcal{L}_1(\mathbf{m})$ and $D = B_Y$ (resp. $\boldsymbol{\omega} = \mathbf{n}$, $f \in \mathcal{L}_1(\mathbf{n})$ and $D = B_Z$). Hence (iii) holds.

Remark 24.3. If $|\alpha(\varphi)| \le M\mu^\bullet(|\varphi|)$ for $\varphi \in \mathcal{K}(T)$, then in the proof of Proposition 4.2 of [T] it is claimed that $\alpha^\bullet \le M\mu^\bullet$. This result requires a proof and is not immediate from the results developed in [T]. Hence there is a laguna in the proof of the said proposition.

Using Theorem 24.2, we now give the vector measure analogues of Theorem 4.1 of [T].

Theorem 24.4. Let X be a Banach space (resp. a quasicomplete lcHs). Let $\mathbf{m} : \mathcal{B}(T) \to X$ (resp. $\mathbf{n} : \delta(\mathcal{C}) \to X$) be σ-additive and $\mathcal{B}(T)$-regular (resp. and $\delta(\mathcal{C})$-regular). Let $(f_\alpha)_{\alpha\in(D,\ge)}$ be an increasing net of non negative lower semicontinuous $\mathbf{m}$-integrable (resp. $\mathbf{n}$-integrable) functions with $f = \sup_\alpha f_\alpha$ also being $\mathbf{m}$-integrable (resp. $\mathbf{n}$-integrable) in T. Then $f_\alpha \to f$ in $\mathcal{L}_1(\mathbf{m})$ (resp. $\mathcal{L}_1(\mathbf{n})$) and consequently, $\lim_\alpha \int_T f_\alpha d\mathbf{m} = \int_T f d\mathbf{m}$ (resp. $\lim_\alpha \int_T f_\alpha d\mathbf{n} = \int_T f d\mathbf{n}$) in X.

Proof. Case 1. X is a Banach space.

Then by Alaoglu's theorem, B_Y and B_Y^+ (resp. B_Z and B_Z^+) are compact in $\sigma(B_Y, \mathcal{L}_1(\mathbf{m}))$ (resp. in $\sigma(B_Z, \mathcal{L}_1(\mathbf{n}))$, where Y, Z, B_Y, B_Y^+, B_Z and B_Z^+ are as in Theorem 24.2. Then, for $\eta \in B_Y^+$ (resp. $\zeta \in B_Z^+$), by Theorem 5.3 of [P2] there exists $\theta \in \mathcal{K}_b(T)^*$ with $\theta \ge 0$ such that $\mu_\theta|_{\mathcal{B}(T)} = \eta$ (resp. by Theorem 4.4 of [P2] there exists $\theta' \in \mathcal{K}(T)^*$ with $\theta' \ge 0$ such that

$\mu_{\theta'}|_{\delta(\mathcal{C})} = \zeta$). Then by Theorem 1, no.1, § 1, Ch. IV of [B],

$$\int_T f d\eta = \sup_\alpha \int_T f_\alpha d\eta = \lim_\alpha \int_T f_\alpha d\eta$$

$$(\text{resp.} \int_T f d\zeta = \sup_\alpha \int_T f_\alpha d\zeta = \lim_\alpha \int_T f_\alpha d\zeta).$$

As f and f_α, $\alpha \in (D, \geq)$, belong to $\mathcal{L}_1(\mathbf{m})$ (resp. $\mathcal{L}_1(\mathbf{n})$), the mappings $\eta \to \int_T f d\eta$ and $\eta \to \int_T f_\alpha d\eta$ (resp. $\zeta \to \int_T f d\zeta$ and $\zeta \to \int_T f_\alpha d\zeta$) are continuous in $\sigma(B_Y^+, \mathcal{L}_1(\mathbf{m}))$ (resp. in $\sigma(B_Z^+, \mathcal{L}_1(\mathbf{n}))$. Consequently, by Dini's lemma, the limit is uniform with respect to $\eta \in B_Y^+$ (resp. $\zeta \in B_Z^+$). Then by (iii) of Theorem 24.2 we have

$$\lim_\alpha \mathbf{m}_1^\bullet(f - f_\alpha, T) = \lim_\alpha \sup_{\eta \in B_Y^+} \int_T |f - f_\alpha| d\eta = 0$$

and hence by (5.3.1) of [P9] we have $\lim_\alpha \int_T f_\alpha d\mathbf{m} = \int_T f d\mathbf{m}$. Similarly, the results for $\mathbf{n}$ are proved.

Case 2. X is a quasicomplete lcHs.

By Theorem 15.13(i) of [P10], $\mathcal{L}_1(\mathbf{m}) = \bigcap_{q \in \Gamma} \mathcal{L}_1(\mathbf{m}_q)$. Then by case 1, $(\mathbf{m}_q)_1^\bullet(f_\alpha - f, T) \to 0$ for $q \in \Gamma$ and hence $f_\alpha \to f$ in $\mathcal{L}_1(\mathbf{m})$. Consequently, by (13.2.1) and Remark 12.5 of [P10], $\int_T f_\alpha d\mathbf{m} \to \int_T f d\mathbf{m}$ as $\alpha \to \infty$. Similarly, the results for $\mathbf{n}$ are proved.

Lemma 24.5. Let X be a Banach space. Let $\mathbf{m} : \mathcal{B}(T) \to X$ (resp. $\mathbf{n} : \delta(\mathcal{C}) \to X$) be σ-additive and Borel regular (resp. and $\delta(\mathcal{C})$-regular). Let $\mathcal{R} = \mathcal{B}(T)$ or $\delta(\mathcal{C})$ and let $\boldsymbol{\omega} = \mathbf{m}$ when $\mathcal{R}=\mathcal{B}(T)$ and $\boldsymbol{\omega} = \mathbf{n}$ when $\mathcal{R}=\delta(\mathcal{C})$. If $\eta \in \mathcal{L}_1(\boldsymbol{\omega})^*$, then, for each bounded Borel function g on T, $g \cdot \eta$ given by

$$(g \cdot \eta)(f) = \int_T f g d\eta \qquad \text{for } f \in \mathcal{L}_1(\boldsymbol{\omega})$$

is well defined and belongs to $\mathcal{L}_1(\boldsymbol{\omega})^*$.

Proof. Let $f \in \mathcal{L}_1(\mathbf{m})$. Then there exist A, N, M such that $N(f) = A \cup N$, $A \in \mathcal{B}(T), N \subset M \in \mathcal{B}(T)$ with $||\mathbf{m}||(M) = 0$. Let $B = A \cup M$. Then $N(f) \subset B \in \mathcal{B}(T)$. If $f \in \mathcal{L}_1(\mathbf{n})$, then by Lemma 23.7 there exists $B \in \mathcal{B}_c(T)$ such that $N(f) \subset B$. By hypothesis, g is a bounded Borel function (resp. $g\chi_B$ is a bounded σ-Borel function) and hence by Theorem 3.5(vi) and Remark 4.3 of [P8], $fg\chi_B \in \mathcal{L}_1(\boldsymbol{\omega})$ and hence

$$(g \cdot \eta)(f) = \int_T f g d\eta = \int_T f g \chi_B d\eta$$

is well defined and $g \cdot \eta$ is a linear functional on $\mathcal{L}_1(\boldsymbol{\omega})$. Let $u_\eta(f) = \int_T f d\eta$ for $f \in \mathcal{L}_1(\boldsymbol{\omega})$ and $\eta \in \mathcal{L}_1(\boldsymbol{\omega})^*$. Then by Lemma 24.1 we have

$$|(g \cdot \eta)(f)| \leq ||g\chi_B||_T \int_T |f| dv(\eta, \mathcal{R}) \leq ||g||_T ||u_\eta|| \boldsymbol{\omega}_1^\bullet(f, T).$$

Hence $g \cdot \eta \in \mathcal{L}_1(\boldsymbol{\omega})^*$.

The following lemmas, which are the same as or similar to Lemmas 23.9, 23.10 and 23.11, are needed to prove the vector measure analogues of Proposition 5.9 of [T].

Lemma 24.6. Let $\mathcal{R}$ and $\boldsymbol{\omega}$ be as in Lemma 24.5. If $(f_n)_1^\infty \subset \mathcal{L}_1(\boldsymbol{\omega})$, there exists a sequence $(K_n)_1^\infty \subset \mathcal{C}$ such that each f_n vanishes $\boldsymbol{\omega}$-a.e. in $T\backslash \bigcup_1^\infty K_k$.

Proof. If $\mathcal{R} = \delta(\mathcal{C})$ and $\boldsymbol{\omega} = \mathbf{n}$, then the result holds by Lemma 23.9. If $\mathcal{R} = \mathcal{B}(T)$ and $\boldsymbol{\omega} = \mathbf{m}$, then by the proof Lemma 24.5 there exist $(B_n)_1^\infty \subset \mathcal{B}(T)$ such that $N(f_n) \subset B_n$ for each n. Let $B = \bigcup_1^\infty B_n$. Then $\bigcup_1^\infty N(f_n) \subset B \in \mathcal{B}(T)$. Then by Theorem 21.1, there exists a sequence $(K_n)_1^\infty \subset \mathcal{C}$ such that $\bigcup_1^\infty K_n \subset B$ and $||\mathbf{m}||(B\backslash \bigcup_1^\infty K_n) = 0$. Since $f_n|_{T\backslash B} = 0$ for each n, it follows that $f_n = 0$ $\mathbf{m}$-a.e. in $T\backslash \bigcup_1^\infty K_n$.

Lemma 24.7. Let X, $\mathcal{R}$, and $\boldsymbol{\omega}$ be as in Lemma 24.5 and let $H = \{x^* \in X^* : |x^*| \leq 1\}$. Given a sequence $(K_n)_1^\infty \subset \mathcal{C}$, there exists a sequence $(x_n^*)_1^\infty \subset H$ such that every set $A \in \sigma(\mathcal{R})$ with $A \subset \bigcup_1^\infty K_n$ is $\boldsymbol{\omega}$-null whenever A is $(x_n^* \circ \boldsymbol{\omega})$-null for each $n \in \mathbb{N}$.

Proof. For each n, $A \cap K_n \in \delta(\mathcal{C})$ by Lemma 18.2 of [P11] whenever $A \in \mathcal{B}(T)$ or $A \in \mathcal{B}_c(T)$. Hence the proof of Lemma 23.10 holds here verbatim in both the cases of $\boldsymbol{\omega}$. Hence the lemma holds.

Lemma 24.8. Let $\mathcal{R} = \mathcal{B}(T)$ or $\delta(\mathcal{C})$. Let $\mu_k : \mathcal{R} \to \mathbb{K}$, $k \in \mathbb{N}$ be σ-additive and $\mathcal{R}$-regular and let $(f_n)_1^\infty \subset \bigcap_{k=1}^\infty \mathcal{L}_1(\mu_k)$. Suppose $(f_n)_1^\infty$ converges weakly to some $h_k \in \mathcal{L}_1(\mu_k)$ for each k. Then there exists a sequence $(g_n)_1^\infty$ such that g_n is a convex combination of $(f_k)_{k\geq n}$ and such that $(g_n)_1^\infty$ converges in mean in $\mathcal{L}_1(\mu_k)$ and also converges pointwise μ_k-a.e. in T for each $k \in \mathbb{N}$.

Proof. The proof of Lemma 23.11 holds here verbatim.

The following theorem gives the vector measure analogues of Proposition 5.9 of [T].

Theorem 24.9. Let X be a Banach space with $c_0 \not\subset X$ and let $\mathbf{m} : \mathcal{B}(T) \to X$ be σ-additive and Borel regular (resp. $\mathbf{n} : \delta(\mathcal{C}) \to X$ be σ-additive and $\delta(\mathcal{C})$-regular). Then $L_1(\mathbf{m})$ (resp. $L_1(\mathbf{n})$) is a weakly sequentially complete Banach space.

Proof. In the light of Theorem 6.8 and Notation 7.6 of [P9], $L_1(\mathbf{m})$ (resp. $L_1(\mathbf{n})$) is a Banach space. To show that these spaces are weakly sequentially complete, let $(f_n)_1^\infty$ be weakly Cauchy in $\mathcal{L}_1(\boldsymbol{\omega})$, where $\boldsymbol{\omega}$ and its domain $\mathcal{R}$ are as in Lemma 24.5. By Lemma 24.6, there exists $(K_n)_1^\infty \subset \mathcal{C}$ such that each f_n vanishes $\boldsymbol{\omega}$-a.e. in $T\backslash \bigcup_1^\infty K_k$. Let $(x_n^*)_1^\infty \subset H = \{x^* \in X^* : |x^*| \leq 1\}$ be chosen as to satisfy the property mentioned in Lemma 24.7 for the sequence $(K_n)_1^\infty$ of compacts. Let $\mu_n = x_n^* \circ \boldsymbol{\omega}$, $n \in \mathbb{N}$. Then by Theorem 3.5(viii) and Remark 4.3 of [P8], $f_n \in \bigcap_{k=1}^\infty L_1(\mu_k)$ for each n.

Let $\eta \in \mathcal{L}_1(\boldsymbol{\omega})^*$. Then by Lemma 24.5, $g \cdot \eta \in \mathcal{L}_1(\boldsymbol{\omega})^*$ for each bounded Borel function g. As $(f_n)_1^\infty$ is weakly Cauchy in $\mathcal{L}_1(\boldsymbol{\omega})$, and as $g \cdot \eta \in \mathcal{L}_1(\boldsymbol{\omega})^*$, $(g \cdot \eta)(f_n))_1^\infty = (\int_T f_n g d\eta)_1^\infty$ is Cauchy in $\mathbb{K}$and hence $(f_n)_1^\infty$ is weakly Cauchy in $\mathcal{L}_1(\eta)$ for each $\eta \in \mathcal{L}_1(\boldsymbol{\omega})^*$. As $\mathcal{L}_1(\eta)$ is weakly sequentially complete, there exists $f_\eta \in \mathcal{L}_1(\eta)$ such that $f_n \to f_\eta$ weakly in $\mathcal{L}_1(\eta)$. On the other hand, by Theorem 24.2, $x_k^* \circ \boldsymbol{\omega} = \mu_k \in \mathcal{L}_1(\boldsymbol{\omega})^*$ (since $x^* \circ \mathbf{m} \in \mathcal{L}_1(\boldsymbol{\omega})^*$ for $x^* \in X^*$ as it is σ-additive and $\mathcal{R}$-regular and $\int_T |f| dv(x^* \circ \boldsymbol{\omega}, \mathcal{R}) \leq ||u_{x^*}|| \boldsymbol{\omega}_1^\bullet(f, T)$, where $u_{x^*}(f) = \int_T f d(x^* \circ \boldsymbol{\omega})$ for $f \in \mathcal{L}_1(\boldsymbol{\omega})$ (see Lemma 24.1)) and taking $\eta = \mu_k$ in the above arguement, there exists f_{μ_k} in $\mathcal{L}_1(\mu_k)$ such that $f_n \to f_{\mu_k}$ weakly in $\mathcal{L}_1(\mu_k)$ for each $k \in \mathbb{N}$. Then by Lemma 24.8 there exists a sequence $(g_n)_1^\infty$ such that each g_n is of the form

$$g_n = \sum_{i=n}^{N(n)} \alpha_i^{(n)} f_i, \ \alpha_i^{(n)} \geq 0, \text{ and } \sum_{i=n}^{N(n)} \alpha_i^{(n)} = 1$$

and such that $(g_n)_1^\infty$ converges in mean in $\mathcal{L}_1(\mu_k)$ and also converges μ_k-a.e. in T for each $k \in \mathbb{N}$. Then by Lemma 24.7, $(g_n)_1^\infty$ converges $\boldsymbol{\omega}$-a.e. in T. Let f be the $\boldsymbol{\omega}$-a.e. pointwise limit of $(g_n)_1^\infty$.

As $f_n \to f_\eta$ weakly in $\mathcal{L}_1(\eta)$, $g_n \to f_\eta$ weakly in $\mathcal{L}_1(\eta)$ for each $\eta \in \mathcal{L}_1(\boldsymbol{\omega})^*$. Then by Theorem 5.3 (resp. by Theorem 4.4) of [P2] and by Theorem 24.2 there exists $\theta \in \mathcal{K}(T)^*$ such that $\eta = \mu_\theta$. Then clearly $\int_A g_n d\mu_\theta$ converges to $\int_A f_\eta d\mu_\theta$ for $A \in \mathcal{B}(T)$. Then by Theorem 23.6, $f = f_\eta$ for each $\eta \in \mathcal{L}_1(\boldsymbol{\omega})^*$. Now, for $x^* \in X^*$, $\eta = x^* \circ \mathbf{m}$ belongs to $\mathcal{L}_1(\boldsymbol{\omega})^*$ by Theorem 24.2 (see the argument given for μ_k in the above). Hence $f \in \mathcal{L}_1(x^* \circ \boldsymbol{\omega})$ for each $x^* \in X^*$. As $c_0 \not\subset X$, by the last part of Theorem 5.8 and by Notation 7.8 of [P9], $f \in \mathcal{L}_1(\boldsymbol{\omega})$. Hence $\mathcal{L}_1(\boldsymbol{\omega})$ is weakly sequentially complete.

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