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"EMBEDDINGS OF L(p,q) SPACES AND ORLICZ SPACES WITH MIXED NORMS"

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"EMBEDDINGS OF $L(p,q)$ SPACES AND ORLICZ SPACES WITH MIXED NORMS"

POR

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1. Introduction. Let $X(0,\infty)$, $Y(0,\infty)$, $Z((0,\infty) \times (0,\infty))$ be r.i. spaces. The problem of finding necessary and sufficient conditions for $X \underset{\pi}{\otimes} Y$ to be continuously embedded into Z is of interest in the theory of integral operators (cf. [7], [2], [4], [5]).

In [7] tensor products of Orlicz spaces and $L(p,q)$ spaces are considered. These results are generalised to the setting of r.i. spaces in [3], [4].

In [4], [6], we consider the problem of embedding r.i. spaces with "mixed norms". Let $X(Y) = \{ f \in M((0,\infty) \times (0,\infty)) : \|f\|_{X(Y)} = \| \, \| f(x,.)\|_Y \|_X < \infty \}$, then

$$X(Y) \subseteq Z \Longrightarrow X \underset{\pi}{\otimes} Y \subseteq Z \tag{1.1}$$

where $\subseteq$ denotes a continous embedding.

The question arises whether the trivial implication (1.1) can be reversed. It is easy to see that if X, Y and Z are L^p spaces then $X(Y) \subseteq Z \Longleftrightarrow X \underset{\pi}{\otimes} Y \subseteq Z$ (See for example [4].)

In [7] the following result is proved for tensor products of $L(p,q)$ spaces,

Theorem A. Let $1 < p < \infty$, $1 \leq q_i \leq \infty$, $i = 1, 2, 3$. Then, $L(p,q_1)(0,\infty) \underset{\pi}{\otimes} L(p,q_2)(0,\infty) \subseteq L(p,q_3)((0,\infty) \times (0,\infty))$ if and only if (i) $\frac{1}{p} + \frac{1}{q_3} \leq \frac{1}{q_1} + \frac{1}{q_2}$, (ii) $\max\{q_1, q_2\} \leq q_3$.

For the Orlicz spaces it is shown in [7] that

Theorem B. Let A, B, C be Young's functions. Then, $L_A(0,\infty) \underset{\pi}{\otimes} L_B(0,\infty) \subseteq L_C((0,\infty) \times (0,\infty))$ if and only if there exists a constant $\theta > 0$ such that

$$A^{-1}(t)\, B^{-1}(s) \leq \theta\, C^{-1}(t.s), \quad \forall\, t, s > 0.$$

In [4], [6] it is shown that

Theorem C. Let $X(0,\infty)$, $Y(0,\infty)$, $Z((0,\infty), (0,\infty))$ be r.i. spaces. Then, the following statements are equivalent

(i) $\Lambda(X) \underset{\pi}{\otimes} M(Y) \subseteq M(Z)$

(ii) $\Lambda(X)\,(M(Y)) \subseteq M(Z)$

(iii) $\exists\, \theta > 0$ such that $\phi_Z(t.s) \leq \theta \phi_X(t)\, \phi_Y(s), \forall\, t, s > 0$

(where $\Lambda(\ )$, $M(\ )$ denote the Lorentz and Marcinkiewicz spaces associated with a r.i. space, and ϕ_X, ϕ_Y, ϕ_Z denote the fundamental functions of X, Y and Z.)

The purpose of this paper is to give necessary and sufficient conditions to embedd Orlicz and Lorentz spaces with mixed norms. Detailed proofs will be given elsewhere.

2. Results. The bassic tools for our research are results in [1], [4] and [6].

Theorem 1. Let $1 < p < \infty$, $1 \leq q_i \leq \infty$, $i = 1, 2, 3$. Then, $L(p,q_1)\,(L(p,q_2)) \subseteq L(p,q_3)\,((0,\infty) \times (0,\infty))$ if and only if (a) $\max\{q_1, q_2\} \leq p \leq q_3$ or (b) $q_1 \leq p$, $q_3 = \infty$.

It follows from Theorem A and Theorem 1 that

Corollary 2. Let $1 < p < \infty$, $1 \leq q \leq \infty$. Then,

$$[L(p,q) \underset{\pi}{\otimes} L(p,q) \subseteq L(p,q)] \Rightarrow [L(p,q)(L(p,q)) \subseteq L(p,q)] \Longleftrightarrow p=q.$$

Theorem 3. Let A, B, C be Young's functions. Then,

$$L_A(0,\infty)(L_B(0,\infty)) \subseteq L_C((0,\infty) \times (0,\infty)) \Longleftrightarrow L_A \underset{\pi}{\otimes} L_B \subseteq L_C.$$

Theorem 4. Let A, B, C, be Young's functions. Then, the following statements are equivalent,

(i) $L_A \underset{\pi}{\otimes} L_B \subseteq L_C$

(ii) $L_A(L_B) \subseteq L_C$

(iii) $L_A(M(L_B)) \subseteq M(L_C)$

(iv) $L_A(\Lambda(L_B)) \subseteq M(L_C)$

(v) $L_A \underset{\pi}{\otimes} M(L_B) \subseteq M(L_C)$

(vi) $L_A \underset{\pi}{\otimes} \Lambda(L_B) \subseteq M(L_C)$

(vii) $\exists\, \theta > 0$ such that $A^{-1}(t)B^{-1}(s) \leq \theta\, C^{-1}(t.s)$, $\forall\, t, s > 0$.

The equivalence (i)$\Longleftrightarrow$(v)$\Longleftrightarrow$(vi)$\Longleftrightarrow$(vii) is due to O'Neil [7], and the relations (ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv)$\Longleftrightarrow$(vii) seem to be new.

3. Outline of Proofs. In [4], [6] it is proved the following

Theorem D. Let $X(0,\infty)$, $Y(0,\infty)$, $Z((0,\infty) \times (0,\infty))$ be r.i. spaces. Then, $X(M(Y)) \subseteq M(Z)$, whenever $\exists\, M > 0$ such that

$$||\phi_{Y'}(|u|)||_{X'} \leq M \,.\, \phi_{Z'}(||u||_1), \quad \forall\, u \in L^1(0,\infty).$$

A simple computation shows that Theorem D implies relations (iii)$\Longleftrightarrow$(vii)$\Longleftrightarrow$(iv), of Theorem 4. The proof of (ii)$\Longleftrightarrow$(vii) is very simple and depends on the fact that $L^1(L^1) \subseteq L^1$. (See [4]).

The sufficiency of the conditions of Theorem 1 follow readily from Theorem D and the L^p embedding theory ([4]).

The necessity of the conditions is more complicated to prove and we rely on some constructions given in [1] and [6].

As an example we give a proof of the following special case of Theorem 1.

<u>Theorem 1'</u>. Let $1 < p < \infty$, $1 \le q_i \le \infty$, $i = 1, 2$. Then, $L(p,q_1)\,(L(p,q_2)) \subseteq L(p,\infty) \Longleftrightarrow q_1 \le p$.

<u>Proof</u>. If $q_1 \le p$, then $L(p,q_1)\,(L(p,q_2)) \subseteq L^p(L(p,q_2)) \subseteq L^p(L(p,\infty)) \subseteq L(p,\infty))$ by Theorem D.

Suppose now that $L(p,q_1)\,(L(p,q_2)) \subseteq L(p,\infty)$. We consider two cases:

(i) $q_2 = \infty$, then by Theorem A we have $q_1 \le p$.

(ii) $q_2 < \infty$.

Suppose that $q_1 > p$, we shall produce $f \in L(p,q_1)\,(L(p,q_2))$ such that $f \notin L(p,\infty)$.

Let $\alpha \in (0,1)$ be such that $p < \alpha q_1$, define

$$f(x,y) = \chi_{(0,(1+x)^{-1})}(y), \min\{1, [\log(x+1)]^{-\frac{\alpha}{p}}\}.$$

Then,

$$f(x,.)^*(t) = \chi_{(0,(1+x)^{-1})}(t).\min\{1,[\log(x+1)]^{-\frac{\alpha}{p}}\}.$$

Thus,

$$\|\ \|f(x,.)\|_{p,q_2}\|_{p,q_1}^{q_1} \sim \int_{e-1}^{\infty}(1+x)^{\frac{-q_1}{p}}[\log(x+1)]^{\frac{-\alpha q_1}{p}} x^{\frac{q_1}{p}} \frac{dx}{x} < \infty.$$

On the other hand,

$$f^*(t) \geq t^{-\alpha/p} \quad \forall t \in (0,\infty)$$

and therefore,

$$\|f\|_{p,\infty} = \sup_{t>0} \{f^*(t)\, t^{1/p}\} \geq \sup_{t>0} \{t^{-\alpha/p}.t^{1/p}\} = \infty.$$

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