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Integral representation for multilinear causal operators

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In this paper, we establish an integral representation theorem for multilinear causal operators in the space of regulated functions. This theorem extends the linear case showed by Hönig[2]. In Viloria[5], we show this theorem for multilinear continuous operators on cartesian products of spaces of regulated functions of a real variable taking values on Banach spaces and in the case when the range is also a regulated functions space.

1 Regulated functions

We consider X, Y, W and Z Banach spaces and $[a,b] \subset \mathbb{R}$ a closed interval.

A **partition** of $[a,b]$ is a finite set $P = \{t_0, \dots, t_n\}$ with $P : a = t_0 < \dots < b = t_n$. We write $n(P) = n$ and $|P| = \max\{t_r - t_{r-1} : 1 \le r \le n(P)\}$. We denoted by $\mathbb{P}[a,b]$ the set of all partitions of $[a,b]$. We write $P_1 \le P_2$ if $P_1 \subset P_2$.

A function $x : [a,b] \to X$ is called a **step function**, and we write $x \in E([a,b], X)$, if there exists a partition $P = \{t_0, \dots, t_n\}$ of $[a,b]$ such that x is constant in each (t_{r-1}, t_r) for every r, $r = 1, \dots, n$.

A function $x : [a,b] \to X$ is called a **regulated function** if it has one-sided limits at every point of $[a,b]$, i.e. if

i) for every $t \in [a,b)$ there exists $x(t^+) = \lim\limits_{\epsilon \downarrow 0} x(t+\epsilon)$ and

ii) for every $t \in (a,b]$ there exists $x(t^-) = \lim\limits_{\epsilon \downarrow 0} x(t-\epsilon)$.

The space of all regulated functions of $[a,b]$ in X is denoted by $G([a,b], X)$.

Theorem 1.1. (Hönig[1], Theorem I.3.1)
Given $x : [a,b] \to X$, the following properties are equivalent
a) $x \in G([a,b], X)$,

b) x is the uniform limit of step functions,
c) for every $\epsilon > 0$ there exists $P \in \mathbb{P}[a,b]$ such that $\omega_{_P}(x) < \epsilon$, where

$$\omega_{_P}(x) = \sup_{1 \leq r \leq n(P)} \ \sup_{t,s \in (t_{r-1},t_r)} ||x(t) - x(s)||.$$

THEOREM 1.2. (HÖNIG[1], THEOREM I.3.6)
$G([a,b],X)$ endowed with the sup *norm is a Banach space.*

DEFINITION 1.1. *A function $x : [a,b] \to X$ is a **left regulated function** if $x(a) = 0$ and $x(t) = x(t^-)$ for every $t \in (a,b]$. In this case we write $x \in G^-([a,b],X)$. This is a closed subspace of $G([a,b],X)$ (* HÖNIG[1], THEOREM I.3.11*).*

DEFINITION 1.2. *$x \in \Omega_0([a,b],X)$ if and only if for every $\epsilon > 0$ the set $\{t \in [a,b] : ||x(t)|| \geq \epsilon\}$ is finite.*
This definition implies that $\Omega_0([a,b],X)$ is a closed subspace of $G([a,b],X)$. In Hönig[2] it is proved that

$$G([a,b],X) = G^-([a,b],X) \oplus \Omega_0([a,b],X).$$

DEFINITION 1.3. *A function $x : [a,b] \to L(W,X)$ is called **simply regulated function** if for every $w \in W$ the function*

$$\begin{aligned} x \cdot w : [a,b] &\longrightarrow X \\ t &\longmapsto x(t)w, \ \textit{is regulated.} \end{aligned}$$

In this case we write $x \in G^\sigma([a,b],L(W,X))$.

$G^\sigma([a,b],L(X,W))$ is a Banach space endowed with the sup norm and $G([a,b],L(W,X)) \subset G^\sigma([a,b],L(W,X))$ (HÖNIG[3], REMARK 1.5).

2 Functions of bounded semi-variation

DEFINITION 2.1. *A **partition of a m-block**, $\displaystyle\prod_{r=1}^{m}[a_r,b_r] \subset \mathbb{R}^m$, is a finite set $P = \displaystyle\prod_{r=1}^{m} P_r$, with $P_r \in \mathbb{P}[a_r,b_r]$, where $a_r = t_{o(r)} < \ldots < t_{n(r)} = b_r$. We set $n(P) = \displaystyle\prod_{r=1}^{m} n(P_r)$ and $|P| = \displaystyle\prod_{r=1}^{m} |P_r|$. We denoted for $\mathbb{P}\left(\displaystyle\prod_{r=1}^{m}[a_r,b_r]\right)$ the set of all partitions of the m-block.*

DEFINITION 2.2. *Let* $z : \prod_{r=1}^{m}[a_r, b_r] \longrightarrow Z$ *and* $P = \prod_{r=1}^{m} P_r$, *with* $P_r \in \mathbb{P}[a_r, b_r]$ *and* $a_r = t_{o(r)} < \ldots < t_{n(r)} = b_r$. *Fixing* r, *we consider an entire* $i(r)$ *with* $1 \leq i(r) \leq n$ *and define*

$$\Delta_{i(r)}z : \prod_{j=1}^{r-1}[a_j, b_j] \times \prod_{j=r+1}^{m}[a_j, b_j] \longrightarrow Z \text{ by}$$

$$(\Delta_{i(r)}z)(s_1, \ldots, s_{r-1}, s_{r+1}, \ldots, s_m) = z(s_1, \ldots, s_{r-1}, t_{i(r)}, s_{r+1}, \ldots, s_m) \\ -z(s_1, \ldots, s_{r-1}, t_{i(r)-1}, s_{r+1}, \ldots, s_m).$$

For $m = 1$, $\Delta_i z = z(t_i) - z(t_{i-1}) \in Z$.
For $m \geq 2$, *we consider* q, $1 \leq q \leq m$; *in this case we can calculate* $\Delta_{i(1)}(\Delta_{i(2)}(\ldots \Delta_{i(q)}z)\ldots)(s_{q+1}, \ldots, s_m)$ *and we denote it by*

$$\Delta_{i(1)}\Delta_{i(2)} \ldots \Delta_{i(q)}z.$$

DEFINITION 2.3. *Let* $X_1, \ldots, X_m$ *and* Y *Banach spaces and consider* $\prod_{r=1}^{m} X_r$ *equipped with the usual product topology induced by the norms on* X_r, $r = 1, \ldots, m$. *If* $x = (x_1, \ldots, x_m) \in \prod_{r=1}^{m} X_r$, $||x|| = \sup ||x_r||$.
$\Lambda : \prod_{r=1}^{m} X_r \longrightarrow Y$ *is called* ***multilinear*** *or* ***m-linear*** *if it is separately linear in each variable. We write* $\Lambda \in L(X_1, \ldots, X_m; Y)$ *if* Λ *is m-linear and continuous (i.e., exist* $M \geq 0$ *such that* $||\Lambda(x_1, \ldots, x_m)|| \leq M||x_1|| \cdots ||x_m||$).

DEFINITION 2.4. *Let* $K : \prod_{r=1}^{m}[a_r, b_r] \longrightarrow Z$. *The* ***Vitali variation*** *of* K *in* $\prod_{r=1}^{m}[a_r, b_r]$ *is given by*

$$V[K] = \sup_P V_P[K],$$

where

$$V_P[K] = \sum_{i(1),\ldots,i(m)}^{n(P)} \big|\,|\Delta_{i(1)} \ldots \Delta_{i(m)}K\big|\,|, P \in \mathbb{P}\bigg(\prod_{r=1}^{m}[a_r, b_r]\bigg).$$

If $V[K] < \infty$ then K is said to be of ***bounded Vitali variation*** *in* $\prod_{r=1}^{m}[a_r, b_r]$ *and we write* $K \in BV\left(\prod_{r=1}^{m}[a_r, b_r], Z\right)$.

DEFINITION 2.5. *Let* $K : \prod_{r=1}^{m}[a_r, b_r] \longrightarrow L(X, Y)$. *The* ***Vitali semi-variation*** *of K in* $\prod_{r=1}^{m}[a_r, b_r]$ *is defined by*

$$SV[K] = \sup_{P} SV_{P}[K],$$

where

$$SV_{P}[K] = \sup_{||x_{i(1)\dots i(m)}|| \leq 1} \left\{ \left|\left| \sum_{i(1),\dots,i(m)}^{n(P)} \Delta_{i(1)} \dots \Delta_{i(m)} K(x_{i(1)\dots i(m)}) \right|\right| : x_{i(1)\dots i(m)} \in X \right\}.$$

If $SV[K] < \infty$ then K is said to be of ***bounded Vitali semi-variation*** *and we write* $K \in SV\left(\prod_{r=1}^{m}[a_r, b_r], L(X, Y)\right)$.

THEOREM 2.1. $BV\left(\prod_{r=1}^{m}[a_r, b_r], L(X, Y)\right) \subset SV\left(\prod_{r=1}^{m}[a_r, b_r], L(X, Y)\right)$ *and if* $K \in BV\left(\prod_{r=1}^{m}[a_r, b_r], L(X, Y)\right)$, *then* $SV[K] \leq V[K]$.

PROOF. Given $K \in BV\left(\prod_{r=1}^{m}[a_r, b_r], L(X, Y)\right)$, $P = \prod_{r=1}^{m} P_r$ with $P_r \in \mathbb{P}[a_r, b_r]$ and $x_{i(1)\dots i(m)} \in X$, with $1 \leq i(r) \leq n(P_r)$, $r = 1, \dots, m$, such that $||x_{i(1)\dots i(m)}|| \leq 1$, then

$$\left|\left| \sum_{i(1),\dots,i(m)}^{n(P)} \Delta_{i(1)} \dots \Delta_{i(m)} K(x_{i(1)\dots i(m)}) \right|\right| \leq \sum_{i(1),\dots,i(m)}^{n(P)} ||\Delta_{i(1)} \dots \Delta_{i(m)} K|| \leq V[K].$$

Hence $SV[K] \leq V[K]$. $\diamond$

DEFINITION 2.6. *For* $K : \prod_{r=1}^{m}[a_r, b_r] \longrightarrow L(X_1, \ldots, X_m; Y)$ *we have the* ***Fréchet semi-variation*** *of* K *in* $\prod_{r=1}^{m}[a_r, b_r]$ *defined by*

$$SF[K] = \sup_P SF_P[K],$$

where

$$SF_P[K] = \sup_{\|x_{i(r)}\| \le 1} \left\{ \left\| \sum_{i(1),\ldots,i(m)}^{n(P)} \Delta_{i(1)} \ldots \Delta_{i(m)} K(x_{i(1)}, \ldots, x_{i(m)}) \right\| : x_{i(r)} \in X_r \right\}$$

If $SF[K] < \infty$ *then* K *is said to be of* ***bounded Fréchet semi-variation*** *and we write* $K \in SF\left(\prod_{r=1}^{m}[a_r, b_r], L(X_1, \ldots, X_m; Y)\right)$.

The next theorem is proved analogously as theorem 2.1.

THEOREM 2.2. $BV\left(\prod_{r=1}^{m}[a_r, b_r], L(X_1, \ldots, X_m; Y)\right) \subset SF\left(\prod_{r=1}^{m}[a_r, b_r], L(X_1, \ldots, X_m; Y)\right)$. *Moreover, if* $K \in BV\left(\prod_{r=1}^{m}[a_r, b_r], L(X_1, \ldots, X_m; Y)\right)$, *then* $SF[K] \le V[K]$.

3 Dushnik interior integral

DEFINITION 3.1. *Let* $e, (e_P)_{P \in \mathbb{P}}$ *being points of a topological space* E. *We write* $e = \lim_P e_P$ *when for all neighborhood* V *of* e *there is* $P_V \in \mathbb{P}$ *such that*

$$P \ge P_V \Rightarrow e_P \in V.$$

DEFINITION 3.2. *Let* $x_r : [a_r, b_r] \longrightarrow X_r;\ r = 1, \ldots, m$ *and* $K : \prod_{r=1}^{m}[a_r, b_r] \longrightarrow L(X_1, \ldots, X_m; Y)$. *If* $\lim_{P \in \mathbb{P}} \sigma_P$ *exists, where* $\mathbb{P} = \mathbb{P}\left(\prod_{r=1}^{m}[a_r, b_r]\right)$ *and* $\sigma_P = \sum_{i(1),\ldots,i(m)}^{n(P)} \Delta_{i(1)} \ldots \Delta_{i(m)} K(x_1(\xi_{i(1)}), \ldots, x_m(\xi_{i(m)}))$ *with* $\xi_{i(r)} \in (t_{i(r)-1} t_{i(r)})$, *then it is called the* ***Dushnik interior integral*** *of the function* $x = (x_1, \ldots, x_m)$ *with respect to the kernel* K *and we denote it by*

$$\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} d_{s_1 \ldots s_m} K(s_1, \ldots, s_m)(x_1(s_1), \ldots, x_m(s_m)).$$

Lemma 3.1. *If* $K \in SF\left(\prod_{r=1}^{m}[a_r, b_r], L(X_1, \ldots, X_m; Y)\right)$ *and* $x_r \in G([a_r, b_r], X_r)$, $r = 1, \ldots, m$, *then,*

i) there exists $\Lambda_K x = \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} d_{s_1 \ldots s_m} K(s_1, \ldots, s_m)(x_1(s_1), \ldots, x_m(s_m))$,

ii) Λ_K *is m-linear,*

iii) $||\Lambda_K x|| \leq SF[K]||x_1|| \cdots ||x_m||$,

iv) if $x_r \in \Omega_0([a_r, b_r], X_r)$ *for any* $r = 1, \ldots, m$, *then* $\Lambda_K x = 0$.

Proof. If $x_r = 0$ for any r or $K = 0$, then the result is immediate. Consider then $x_r \neq 0$, $r = 1, \ldots, m$, and $K \neq 0$.

i) We will show that the Cauchy criterion holds.

Let $\epsilon > 0$, then for every r, $r = 1, \ldots, m$, according to Theorem 1.1, there exists $P_r(\epsilon) \in \mathbb{P}[a_r, b_r]$ such that

$$\omega_{P_r(\epsilon)}(x_r) < \frac{\epsilon||x_r||}{2SF[K]||x_1|| \cdots ||x_m||}.$$

If $P \geq P(\epsilon) = \prod_{r=1}^{m} P_r(\epsilon)$, we can obtain P from $P(\epsilon)$ by inserting a finite number of points in the partitions $P_r(\epsilon)$. By induction, we are thus reduced to the case when P is obtained inserting one point in some partition $P_k(\epsilon)$ for some $k, k = 1, \ldots, m$. Let $\mathcal{O}_k$ the point considered in some interval of $P_k(\epsilon)$. Thus

$$\begin{aligned}
\sigma_P - \sigma_{P(\epsilon)} &= \sum_{i(1),\ldots,i(m)}^{n(P)} \Delta_{i(1)} \ldots \Delta_{i(m)} K(x_1(\xi_{i(1)}), \ldots, x_{k-1}(\xi_{i(k-1)}), \\
& \quad x_k(\xi_{i(k)}) - x_k(\xi_{\mathcal{O}_k}),\ x_{k+1}(\xi_{i(k+1)}), \ldots, x_m(\xi_{i(m)})) \\
&= \textstyle\sum_{i(1),\ldots,i(m)}^{n(P)} \Delta_{i(1)} \ldots \Delta_{i(m)} K\Bigg(\frac{x_1(\xi_{i(1)})}{||x_1||}, \ldots, \\
& \quad \frac{2SF[K]||x_1||\cdots||x_m||}{\epsilon||x_k||}(x_k(\xi_{i(k)} - x_k(\xi_{\mathcal{O}_k})), \ldots, \frac{x_m(\xi_{i(m)})}{||x_m||}\Bigg)\frac{\epsilon}{2SF[K]}.
\end{aligned}$$

This yields

$$||\sigma_P - \sigma_{P(\epsilon)}|| \leq SF[K]\frac{\epsilon}{2SF[K]} = \frac{\epsilon}{2}.$$

Consequently

$$P, \bar{P} \geq P(\epsilon) \Longrightarrow ||\sigma_{_P} - \sigma_{_{\bar{P}}}|| \leq \epsilon.$$

ii) Can be proved directly by the definition.

iii) For any $P \in \mathbb{P}\left(\prod_{r=1}^{m}[a_r, b_r]\right)$ we have

$$||\sigma_{_P}|| = \left|\left|\sum_{i(1),\ldots,i(m)}^{n(P)} \Delta_{i(1)} \ldots \Delta_{i(m)} K\left(\frac{x_1(\xi_{i(1)})}{||x_1||}, \ldots, \frac{x_m(\xi_{i(m)})}{||x_m||}\right)\right|\right| \cdot ||x_1|| \cdots ||x_m||$$
$$\leq SF[K]||x_1|| \cdots ||x_m||.$$

Thus, taking the limit, it results

$$||\Lambda_{_K} x|| \leq SF[K]||x_1|| \cdots ||x_m||.$$

iv) Without loss of generality, we can consider $x_1 \in \Omega_0([a_1, b_1], X_1)$. Then by the definition of $\Omega_0, \forall\epsilon > 0$ there exists $P_1(\epsilon) \in \mathbb{P}[a_1, b_1]$ such that

$$\{t \in [a_1, b_1] : ||x_1(t)|| \geq \epsilon/SF[K]||x_2|| \cdots ||x_m||\} \subset P_1(\epsilon).$$

Hence, if $P = \prod_{r=1}^{m} P_r$ with $P_1 \geq P_1(\epsilon)$ and the other partitions being arbitraries, we get

$$||\sigma_{_P}|| = \left|\left|\sum_{i(1),\ldots,i(m)}^{n(P)} \Delta_{i(1)} \ldots \Delta_{i(m)} K\left(\frac{SF[K]||x_2|| \cdots |x_m||}{\epsilon} x_1(\xi_{i(1)}),\right.\right.\right.$$
$$\left.\left.\left.\frac{x_2(\xi_{i(2)})}{||x_2||}, \ldots, \frac{x_m(\xi_{i(m))}}{||x_m||}\right)\right|\right| \frac{\epsilon}{SF[K]} < \epsilon.$$

Then $\Lambda_{_K}\ x = 0$. $\diamond$

Such as in the bilinear case (Prandini[4], Theorem 4.3) the following result holds

THEOREM 3.1. *If* $K \in SF\left(\prod_{r=1}^{m}[a_r, b_r], L(X_1, \ldots, X_m; Y)\right)$ *and* $x_r \in G([a_r, b_r], X_r)$ *for every* $r, r = 1, \ldots, m$. *Then*

$$\Lambda_K x = \int_{a_m}^{b_m} d_{s_m} \ldots \int_{a_1}^{b_1} d_{s_1} K(s_1, \ldots, s_m) x_1(s_1) \cdots x_m(s_m).$$

4 Integral representation

DEFINITION 4.1. *Let* $K : \prod_{r=1}^{m}[a_r, b_r], \longrightarrow L(X_1, \ldots, X_m; Z)$ *we write* $K \in SF_{a^m}\left(\prod_{r=1}^{m}[a_r, b_r], L(X_1, \ldots, X_m; Z)\right)$ *when*

$$K(s_1, \ldots, s_{i-1}, a_i, s_{i+1}, \ldots, s_m) = 0 \;\; \forall i, i = 1, \ldots, m.$$

THEOREM 4.1. *The mapping* $K \longmapsto \Lambda_K$, *where*

$$\Lambda_K x = \int_{a_m}^{b_m} d_{s_m} \ldots \int_{a_1}^{b_1} d_{s_1} K(s_1, \ldots, s_m) x_1(s_1) \cdots x_m(s_m)$$

is an isometry between the Banach spaces

$$SF_{a^m}\left(\prod_{r=1}^{m}[a_r, b_r], L(X_1, \ldots, X_m; Z)\right)$$

and

$$L(G^-([a_1, b_1], X_1), \ldots, G^-([a_m, b_m], X_m); Z),$$

moreover

$$K(s_1, \ldots, s_m)(\bar{x}_1, \ldots, \bar{x}_m) = \Lambda_K(\chi_{(a_1, s_1]}\bar{x}_1, \ldots, \chi_{(a_m, s_m]}\bar{x}_m)$$

and

$$||\Lambda_K|| = SF[K].$$

PROOF. By Lemma 3.1 the mapping is well defined, linear and continuous, and we have $||\Lambda|| \leq SF[K]$.
Injectivity: If $K \neq 0$ there are $\tau_r \in (a_r, b_r]$ and $\bar{x}_r \in X_r, r = 1, \dots, m$ such that

$$K(\tau_1, \dots, \tau_m)(\bar{x}_1, \dots, \bar{x}_m) \neq 0.$$

Let $x_r = X_{(a_r,\tau_r]}\bar{x}_r \in G^-([a_r, b_r], X_r)$, then $\Lambda_K \neq 0$, since

$$\begin{aligned}\Lambda_K x &= \int_{a_m}^{b_m} \dots \int_{a_1}^{b_1} d_{s_1 \dots s_m} K(s_1, \dots, s_m)(\chi_{(a_1,\tau_1]}(s_1)\bar{x}_1, \dots, \chi_{(a_m,\tau_m]}(s_m)\bar{x}_m) \\ &= K(\tau_1, \dots, \tau_m)(\bar{x}_1, \dots, \bar{x}_m).\end{aligned}$$

Surjectivity: Give $\Lambda \in L(G^-[a_1, b_1], X_1), \dots, G^-([a_m, b_m], X_m)$, if there exists $K \in SF_{a^m}\left(\prod_{r=1}^{m}[a_r, b_r], L(X_1, \dots, X_m; Y)\right)$ such that $\Lambda = \Lambda_K$ then

$$K(\tau_1, \dots, \tau_m)(\bar{x}_1, \dots, \bar{x}_m) = \Lambda(\chi_{(a_1,\tau_1]}\bar{x}_1, \dots, \chi_{(a_m,\tau_m]}\bar{x}_m),$$

$\tau_r \in (a_r, b_r]$ and $\bar{x}_r \in X_r, r = 1, \dots, m$. We are taking this as a definition of K. To end the proof we must show a) $SF[K] \leq ||\Lambda||$ and b) $\Lambda_K = \Lambda$.

a) $SF_P[K] = \sup_{||\bar{x}_{i(r)}|| \leq 1} \left\{ \left|\left| \sum_{i(1),\dots,i(m)}^{n(p)} \Delta_{i(1)} \dots \Delta_{i(m)} K(\bar{x}_{i(1)}, \dots, \bar{x}_{i(m)}) \right|\right| : \bar{x}_{i(r)} \in X_r \right\}$

$$\begin{aligned}&= \sup_{||\bar{x}_{i(r)}|| \leq 1} \left\{ \left|\left| \sum_{i(1),\dots,i(m)}^{n(p)} \Lambda(\chi_{(t_{i(1)-1},t_{i(1)}]}\bar{x}_{i(1)}, \dots, \chi_{(t_{i(m)-1},t_{i(m)}]}\bar{x}_{i(m)}) \right|\right| \right\} \\ &= \sup_{||\bar{x}_{i(r)}|| \leq 1} \left\{ \left|\left| \Lambda\left(\sum_{i(1)=1}^{n(p_1)} \chi_{(t_{i(1)-1},t_{i(1)}]}\bar{x}_{i(1)}, \dots, \sum_{i(m)=1}^{n(p_m)} \chi_{(t_{i(m)-1},t_{i(m)}]}\bar{x}_{i(m)} \right) \right|\right| \right\} \\ &\leq ||\Lambda||.\end{aligned}$$

b) We have $\Lambda, \Lambda_K \in L(G^-([a_1, b_1], X_1), \dots, G^-([a_m, b_m], X_m); Y)$. In order to show the equality $\Lambda_K = \Lambda$ it is enough to prove that they coincide on the elements of the form $\chi_{(a_1,\tau_1]}\bar{x}_1, \dots, \chi_{(a_m,\tau_m]}\bar{x}_m$ since these elements form a total set in $G^-([a_1, b_1], X_1), \dots, G^-([a_m, b_m], X_m)$, respectively. Indeed

$$\begin{aligned}&\Lambda_K(\chi_{(a_1,\tau_1]}\bar{x}_1, \dots, \chi_{(a_m,\tau_m]}\bar{x}_m) = \\ &= \int_{a_m}^{b_m} \dots \int_{a_1}^{b_m} d_{s_1 \dots s_m} K(s_1 \dots s_m)(\chi_{(a_1,\tau_1]}(s_1)\bar{x}_1, \dots, \chi_{(a_m,\tau_m]}(s_m)\bar{x}_m)\end{aligned}$$

$$= K(\tau_1, .., \tau_m)(\bar{x}_1, \ldots, \bar{x}_m)$$
$$= \Lambda(\chi_{(a_1,\tau_1]}\bar{x}_1, \ldots, \chi_{(a_m,\tau_m]}\bar{x}_m). \qquad \diamond$$

Consider now the case for operators between function spaces.

DEFINITION 4.2. *Let* $K : [a,b] \times \prod_{r=1}^{m} [a_r, b_r] \longrightarrow L(X_1, \ldots, X_m; Y)$. *We define* $K^t : \prod_{r=1}^{m} [a_r, b_r] \longrightarrow L(X_1, \ldots, X_m; Y)$ *and* $K_{s^m} : [a,b] \longrightarrow L(X_1, \ldots, X_m; Y)$ *as being*

$$K^t(s_1, \ldots, s_m) = K(t, s_1, \ldots, s_m) = K_{s^m}(t).$$

Also, consider the following properties:
(G^σ) : K is ***simply regulated as a function of*** t, i.e.

$$K_{s^m} \in G^\sigma([a,b], L(X_1, \ldots, X_m; Y)).$$

(SF^u) : K is ***uniformly of bounded Fréchet semi-variation as a function of*** $(s_1, \ldots, s_m)$, i.e. $SF^u[K] = \sup_{t\in[a,b]} SF[K^t] < \infty$.
$(SF^u_{a^m})$: K satisfies (SF^u) and $K(t, a_1, \ldots, a_m) = 0 \ \forall t \in [a,b]$.
When K verify both (G^σ) and (SF^u), then we write

$$K \in G^\sigma \cdot SF^u\left([a,b] \times \prod_{r=1}^{m} [a_r, b_r], L(X_1, \ldots, X_m; Y)\right).$$

Analogously when $K \in G^\sigma \cdot SF^u_{a^m}$.

THEOREM 4.2. *The mapping* $K \longmapsto \Lambda_K$, *for which*

$$\Lambda_K(x_1, \ldots, x_m)(t) = \int_{a_m}^{b_m} d_{s_m} \ldots \int_{a_1}^{b_1} d_{s_1} K(t, s_1, \ldots, s_m) x_1(s_1) \cdots x_m(s_m)$$

is an isometry between the Banach spaces

$$G^\sigma \cdot SF^u_{a^m}\left([a,b] \times \prod_{r=1}^{m} [a_r, b_r], L(X_1, \ldots, X_m; Y)\right)$$

and

$$L(G^-([a_1,b_1],X_1),\dots,G^-([a_m,b_m],X_m);G([a,b],Y)),$$

putting $K(t,s_1,\dots,s_m)(\bar{x}_1,\dots,\bar{x}_m)=\Lambda_K\ (\chi_{(a_1,s_1]}\bar{x}_1,\dots,\chi_{(a_m,s_m]}\bar{x}_m)(t)$*. We have furthermore,* $||\Lambda_K||=SF^u[K]$*.*

PROOF. For $t\in[a,b], K_{s^m}$ is of bounded Fréchet semi-variation; by other hand x_r is regulated, for $r=1,\dots m$, then $\Lambda_K(x_1,\dots,x_m)(t)$ is well defined. Analogously, as in the previous theorem linearity and injectivity are direct consequences of the definition. Again for $t\in[a,b]$,

$$||(\Lambda_K x)(t)||\le SF[K^t]||x_1||\cdots||x_m||.$$

Hence

$$||\Lambda_K x||\le SF^u[K]||x_1||\cdots||x_m||.$$

Then

$$||\Lambda_K||\le SF^u[K].$$

Surjectivity: Let $\Lambda\in L(G^-([a_1,b_1],X_1),\dots,G^-([a_m,b_m],X_m);G^-([a,b],Y))$. By the previous theorem there exists

$$\bar{K}\in SF_{a^m}\left(\prod_{r=1}^{m}[a_r,b_r],L(X_1,\dots,X_m;G([a,b],Y)\right)$$

such that

$$\Lambda x=\int_{a_m}^{b_m}d_{s_m}\dots\int_{a_1}^{b_1}d_{s_1}\bar{K}(s_1,\dots,s_m)x_1(s_1)\dots x_m(s_m),$$

when

$$\bar{K}(s_1,\dots,s_m)(\bar{x}_1,\dots,\bar{x}_m)=\Lambda(\chi_{(a_1,s_1]}\bar{x}_1,\dots,\chi_{(a_m,s_m]}\bar{x}_m).$$

Defining, then,

$$K:[a,b]\times\prod_{r=1}^{m}[a_r,b_r]\longrightarrow L(X_1,\dots,X_m;Y)$$

as

$$K(t,s_1,\ldots,s_m)(\bar{x}_1,\ldots,\bar{x}_m) = (\bar{K}(s_1,\ldots,s_m)(\bar{x}_1,\ldots,\bar{x}_m))(t),$$

we have

$$\Lambda(\chi_{(a_1,s_1]}\bar{x}_1,\ldots,\chi_{(a_m,s_m]}\bar{x}_m)(t) = K(t,s_1,\ldots,s_m)(\bar{x}_1,\ldots,\bar{x}_m),$$

and again

$$(\Lambda x)(t) = \int_{a_m}^{b_m} d_{s_m} \ldots \int_{a_1}^{b_1} d_{s_1} K(t,s_1,\ldots,s_m) x_1(s_1)\cdots x_m(s_m).$$

Now we will prove that $K \in G^\sigma \cdot SF^u_{a^m}\left([a,b]\times\prod_{r=1}^{m}[a_r,b_r], L(X_1,\ldots,X_m;Y)\right)$.

a) $K(t,a_1,\ldots,a_m)(\bar{x}_1,\ldots,\bar{x}_m) = (\bar{K}(a_1,\ldots,a_m)(\bar{x}_1,\ldots,\bar{x}_m))(t) = 0.$

b) K is uniformy of bounded Fréchet semi-variation in $(s_1,\ldots,s_m)$. Indeed, let $P = \prod_{r=1}^{m} P_r, P_r \in \mathbb{P}[a_r,b_r]$ and $\bar{x}_{i(r)} \in X_r$ with $||\bar{x}_{i(r)}|| \leq 1\ \forall i(r), 1 \leq i(r) \leq n(P_r), \forall r = 1,\ldots,m$. Then

$$\left|\left|\sum_{i(1),\ldots,i(m)}^{n(P)} \Delta_{i(1)}\ldots\Delta_{i(m)} K(t)(\bar{x}_{i(1)},\ldots,\bar{x}_{i(m)})\right|\right| =$$

$$= \left|\left|\Lambda\Big(\sum_{i(1)=1}^{n(P_1)} \chi_{(t_{i(1)-1},t_{i(1)}]}\bar{x}_{i(1)},\ldots,\sum_{i(1)=1}^{n(P_m)} \chi_{(t_{i(m)-1},t_{i(m)}]}\bar{x}_{i(m)}\Big)(t)\right|\right|$$

$$\leq \left|\left|\Lambda\left(\sum_{i(1)=1}^{n(P_1)} \chi_{(t_{i(1)-1},t_{i(1)}]}\bar{x}_{i(1)},\ldots,\sum_{i(1)=1}^{n(P_m)} \chi_{(t_{i(m)-1},t_{i(m)}]}\bar{x}_{i(m)}\right)\right|\right|$$

$$\leq ||\Lambda|| \left|\left|\sum_{i(1)=1}^{n(P_1)} \chi_{(t_{i(1)-1},t_{i(1)}]}\bar{x}_{i(1)}\right|\right| \cdots \left|\left|\sum_{i(1)=1}^{n(P_m)} \chi_{(t_{i(m)-1},t_{i(m)}]}\bar{x}_{i(m)}\right|\right| \leq ||\Lambda||.$$

Hence

$$SF[K^t] \leq ||\Lambda|| \ \ \forall t \in [a,b].$$

And, then,

$$SF^u[K] \leq ||\Lambda||.$$

c) K is simply regulated as function of the variable t; since Λ takes values in $G([a,b],Y)$ according to the definition of K, we have that, for every $(s_1,\ldots,s_m)$, the function

$$\phi : [a,b] \longrightarrow Y$$

defined by

$$\phi(t) = K(t, s_1, \ldots, s_m)(\bar{x}_1, \ldots, \bar{x}_m),$$

for all $(\bar{x}_1, \ldots, \bar{x}_m)$, is regulated.

$\diamond$

For close this section we treat the causal operators case.

Definition 4.3. *Let* $K \in G^\sigma \cdot SF^u([a,b]^{m+1}, L_m(X,Y))$ *where* $[a,b]^{m+1} = \prod_{r=1}^{m+1} [a,b]$ *and* $L_m(X;Y) = L(X,\ldots,X;Y)$. *If for every* $x \in X$ *the function* $K_\Delta : [a,b] \longrightarrow Y$ *defined by* $t \longmapsto K(t,\ldots,t)(x,\ldots,x)$ *is regulated, we say that* K *is* ***simply regulated on the diagonal*** *and we denote this by*

$$K \in G^\sigma_\Delta \cdot SF^u([a,b]^{m+1}, L_m(X;Y)).$$

If in addition $K_\Delta(t) = 0$, *for every* $t \in [a,b]$ *then we say that* K ***vanish in the diagonal*** *and we write*

$$K \in G^\sigma_\circ \cdot SF^u([a,b]^{m+1}, L_m(X;Y)).$$

Definition 4.4. $P \in L_m(G([a,b],X); G([a,b],Y))$ *is a* ***causal operador*** *if for every* $x \in G([a,b],X)$ *and for every* $T \in [a,b]$,

$$x_{|_{[a,T]}} = 0 \Rightarrow P(x,\ldots,x)_{|_{[a,T]}} = 0.$$

Definition 4.5. *Let* $K \in G^{\sigma}\cdot SF^{u}([a,b]^{m+1}, L_m(X;Y))$. *For* $x = (x_1,\ldots,x_m)$ *with* $x_r \in G^{-}([a,b],X), r = 1,\ldots,m,$ *we define*

$$(kx)(t) = \int_a^t d_{s_m} \ldots \int_a^t d_{s_1} K(t,s_1,\ldots,s_m)x_1(s_1)\cdots x(s_m),$$

for $t \in [a,b]$.

Now we prove that the causal multilinear operators are also representables.

Theorem 4.3. *The mapping* $K \longmapsto k$ *is an isometry between the Banach space* $G^{\sigma}_{\circ}\cdot SF^{u}([a,b]^{m+1}, L_m(X;Y))$ *and the subspace of the causal operators of* $L_m(G^{-}([a,b],X); G([a,b],Y))$, *where* $||k|| = SF^{u}[K]$ *and* $K(t,s_1,\ldots,s_m)$ $(\bar{x}_1,\ldots,\bar{x}_m) = k(\chi_{(a,t]}\bar{x}_1,\ldots,\chi_{(a,t]}\bar{x}_m)(t)$.

Proof. Given $K \in G^{\sigma}\cdot SF^{u}([a,b]^{m+1}, L_m(X;Y))$ we define

$$\tilde{K}(t,s_1,\ldots,s_m) = \begin{cases} K(t,s_1,\ldots,s_m) & \text{if} \quad t \geq s_r, \text{ for every } r,\ r = 1,\ldots,m \\ \\ K(t,t,\ldots,t) & \text{if} \quad t < s_r, \text{ for some } r,\ r = 1,\ldots,m. \end{cases}$$

If $K \in G^{\sigma}_{\Delta}\cdot SF^{u}([a,b]^{m+1}, L_m(X;Y))$, then $\tilde{K} \in G^{\sigma}_{\Delta}.SF^{u}([a,b]^{m+1}, L_m(X;Y))$. In addition, we have

$$\begin{aligned}(kx)(t) &= \int_a^t d_{s_m} \ldots \int_a^t d_{s_1} K(t,s_1,\ldots,s_m)x_1(s_1)\ldots x_m(s_m) \\ &= \int_a^b d_{s_m} \ldots \int_a^b d_{s_1} \tilde{K}(t,s_1,\ldots,s_m)x_1(s_1)\ldots x_m(s_m).\end{aligned}$$

As in the previous theorem we have

$$kx \in G([a,b],Y), x = (x_1,..,x_m)$$

with

$$x_r \in G([a,b],X), r = 1,\ldots,m.$$

Reciprocally, if $\Lambda \in L_m(G^{-}([a,b],X); G([a,b],Y))$ is a causal operator then, by the previous theorem, there exits $\bar{K} \in G^{\sigma}\cdot SF^{u}_{a}([a,b]^{m+1}, L_m(X;Y))$ such that

$$(\Lambda x)(t) = \int_a^b d_{\tau_m} \dots \int_a^b d_{\tau_1} \bar{K}(t, \tau_1, \dots, \tau_m) x_1(\tau_1) \cdots x_m(\tau_m)$$

where $\bar{K}(t, \tau_1, \dots, \tau_m) = \Lambda(\chi_{(a_1,\tau_1]}\bar{x}_1, \dots, \chi_{(a_m,\tau_m]}\bar{x}_m)(t),\ t \in [a,b],\ \bar{x}_r \in X$ and $\tau_r \in (a,b], r = 1, \dots, m$.

Since Λ is causal, for $s_r \geq t, r = 1, \dots, m$, we have

$$\Lambda(\chi_{(a,s_1]}\bar{x}_1, \dots, \chi_{(a,s_m]}\bar{x}_m)(t) = \Lambda(\chi_{(a,t]}\bar{x}_1, \dots, \chi_{(a,t]}\bar{x}_m)(t),$$

i.e.

$$\bar{K}(t, s_1, \dots, s_m)(\bar{x}_1, \dots, \bar{x}_m) = \bar{K}(t, \dots, t)(\bar{x}_1, \dots, \bar{x}_m).$$

Once more:

$$(\Lambda x)(t) = \int_a^t d_{s_m} \dots \int_a^t d_{s_1} \bar{K}(t, s_1, \dots, s_m) x_1(s_1) \cdots x_m(s_m).$$

If $x_r(s_r) = \bar{c}_r$, we have

$$\begin{aligned}(\Lambda x)(t) &= \textstyle\int_a^t d_{s_m} \dots \int_a^t d_{s_1} \bar{K}(t, s_1, \dots, s_m)\bar{c}_1 \dots \bar{c}_m \\ &= \bar{K}(t, \dots, t) + \sum_{s_i \in \{t,a\}}^{2^m} (-1)^{\#(a)} \bar{K}(t, s_1, \dots, s_m)\bar{c}_1 \dots \bar{c}_m,\end{aligned}$$

where $\#(a) = \sum_{i=1}^m \delta_i$, with $\delta_i = \begin{cases} 1 & \text{if} \quad s_i = a, \\ 0 & \text{if} \quad s_i = t. \end{cases}$

Since Λx is simply regulated and $K \in G^\sigma \cdot SF^u$ then the mapping $t \longmapsto \bar{K}(t, s, \dots, s)$ is regulated and hence the mapping $t \longmapsto \bar{K}(t, \dots, t)$ is regulated. Then $\bar{K} \in G^\sigma_\Delta \cdot SF^u([a,b]^{m+1}, L_m(X;Y))$.

Let $K(t, s_1, \dots, s_m) = \bar{K}(t, s_1, \dots, s_m) - \bar{K}(t, t, \dots, t)$. Then
$\bar{K} \in G^\sigma_\Delta \cdot SF^u$ and $K(t, \dots, t) = 0$ for every $t \in [a,b]$. Hence $K_\Delta(t) = 0$ for every $t \in [a,b]$ and

$$K \in G^\sigma_\circ \cdot SF^n$$

and

$$(\Lambda x)(t) = \int_a^t d_{s_m} \dots \int_a^t d_{s_1} K(t, s_1, \dots, s_m) x_1(s_1) \cdots x_m(s_m).$$

$\diamond$

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