Revista Notas de Matemática
Vol.4(2), No. 269, 2008, pp. 66-78
http://www.saber.ula.ve/notasdematematica/
Comisión de Publicaciones
Departamento de Matemáticas
Facultad de Ciencias
Universidad de Los Andes

Cone metric spaces and fixed point theorems of $T-$contrative mappings

José R. Morales and Edixón Rojas

Resumen

En este trabajo, se estudia la existencia de puntos fijos para las asignaciones definidas en total, (secuencialmente compacto) cono espacio métrico, (M, d) que satisface una desigualdad de contracción general de depender de otra función

Palabras claves: Cono espacios métricos, punto fijo, el mapeo de contracción, de forma secuencial convergente.

Abstract

In this paper, we study the existence of fixed points for mappings defined on complete, (sequentially compact) cone metric space, (M, d) satisfying a general contractive inequality depend on another function

key words. Cone metric spaces, fixed point, contractive mapping, sequentially convergent
AMS(MOS) subject classifications. 46J10, 46J15, 47H10.

1 Introduction

The concept of cone metric space was introduced by Huan Long - Guang and Zhang Xian [2], where the set of real numbers is replaced by an ordered Banach space. They introduced the basic definitions and discuss some properties of convergence of sequences in cone metric spaces.

They also obtained various fixed point theorems for contractive single - valued maps in such spaces. Subsequently, some other mathematicians, ([4],[5],[6],...), have generalized the results of Guang and Zhang [2].

Recently, A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh [1] introduced a new class of contractive mappings: $T-$contraction and $T-$contrative extending the Banach's contraction principle and the Edelstein's fixed point theorem, (see [3]) respectively.

The purpose of this paper is to analyze the existence of fixed points for a self-map S defined on a complete, (sequentially compact) cone metric space, (M, d) satisfying the $T-$contraction and $T-$contractive condition.

Our results extend some fixed points theorems of [1] and [2].

2 Preliminary facts

Consistent with Guang and Zhang [2], we recall the definitions of cone metric space, the notion of convergence and other results that will be needed in the sequel.

Let E be a real Banach space and P a subset of E. P is called a *cone* if and only if:

P1.- P is nonempty, closed and $P \neq \{0\}$;

P2.- $a, b \in \mathbb{R},\ a, b \geq 0$ and $x, y \in P \Rightarrow ax + by \in P$;

P3.- $x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ on E with respect to P by

$$x \leq y \quad \text{,if and only if} \quad y - x \in P.$$

We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stands for $y - x \in \operatorname{Int} P$, where $\operatorname{int} P$ denotes the interior of P. The cone $P \subset E$ is called *normal* if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y, \quad \text{implies} \quad \|x\| \leq K\|y\|.$$

The least positive number satisfying inequality above is called the *normal constant* of P.

The cone P is called *regular* if every increasing sequence which is bounded from above is convergent. That is, if (x_n) is a sequence such that

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \leq y$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \longrightarrow 0,\ (n \to \infty)$.

In the following we always suppose E is a Banach space, P is a cone with $\operatorname{int} P \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 2.1 ([2]) *Let M be a nonempty set. Suppose the mapping $d : M \times M \longrightarrow E$ satisfies:*

d1.- $0 < d(x, y)$ *for all* $x, y \in M$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

d2.- $d(x, y) = d(y, x)$ *for all* $x, y \in M$;

d3.- $d(x, y) \leq d(x, z) + d(y, z)$ *for all* $x, y, z \in M$.

Then d is called a cone metric *on M and (M, d) is called a* cone metric space.

It is obvious that cone metric spaces generalize metric spaces.

Example 2.2 1. ([2, Example 1]) *Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2$, $M = \mathbb{R}$ and $d : M \times M \longrightarrow E$ such that*

$$d(x, y) = \Big(|x - y|,\ \alpha|x - y|\Big)$$

where $\alpha \geq 0$ is a constant. Then (M, d) is a cone metric space.

2. *Let $E = (C_{[0,1]}, \mathbb{R})$, $P = \{\varphi \in E : \varphi \geq 0\} \subset E$, $M = \mathbb{R}$ and $d : M \times M \longrightarrow E$ such that*

$$d(x, y) = |x - y|\varphi$$

where $\varphi(t) = e^t \in E$. Then (M, d) is a cone metric space.

Definition 2.3 ([2]) *Let (M, d) be a cone metric space. Let (x_n) be a sequence in M. Then:*

(i) *(x_n) converges to $x \in M$ if, for every $c \in E$, with $0 \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

$$d(x_n, x) \ll c.$$

We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \longrightarrow x$, $(n \to \infty)$.

(ii) *If for any $c \in E$, there is a number $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$*

$$d(x_n, x_m) \ll c,$$

then (x_n) is called a Cauchy sequence *in M;*

(iii) *(M, d) is a* complete cone metric space *if every Cauchy sequence is convergent in M.*

The following lemma will be useful for us to prove our main results.

Lemma 2.4 ([2]) *Let (M, d) be a cone metric space, P a normal cone with normal constant K and (x_n) is a sequence in M.*

(i) *(x_n) converges a $x \in M$ if and only if*

$$\lim_{n\to\infty} d(x_n, x) = 0;$$

(ii) *If (x_n) is convergent then it is a Cauchy sequence;*

(iii) *(x_n) is a Cauchy sequence if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$;*

(iv) *If $x_n \longrightarrow x$ and $x_n \longrightarrow y$, $(n \to \infty)$ then $x = y$;*

(v) *If $x_n \longrightarrow x$ and (y_n) is another sequence in M such that $y_n \longrightarrow y$, then $d(x_n, y_n) \longrightarrow d(x, y)$.*

Definition 2.5 *Let (M, d) be a cone metric space. It for any sequence (x_n) in M, there is a subsequence (x_{n_i}) of (x_n) such that (x_{n_i}) is convergent in M. Then M is called a* sequentially compact cone metric space.

Next Definition and subsequent Lemma are given in [1] in the scope of metric spaces, here we will rewrite it in terms of cone metric spaces.

Definition 2.6 *Let (M, d) be a cone metric space, P a normal cone with normal constant K and $T : M \longrightarrow M$. Then*

(i) *T is said to be* continuous *if $\lim_{n\to\infty} x_n = x$, implies that $\lim_{n\to\infty} Tx_n = Tx$ for every (x_n) in M;*

(ii) *T is said to be* sequentially convergent *if we have, for every sequence (y_n), if $T(y_n)$ is convergent, then (y_n) also is convergent;*

(iii) *T is said to be* subsequentially convergent *if we have, for every sequence (y_n), if $T(y_n)$ is convergent, then (y_n) also is convergent.*

Lemma 2.7 *If (M, d) be a sequence compact cone metric space, then every function $T : M \longrightarrow M$ is subsequentially convergent and every continuous function $T : M \longrightarrow M$ is sequentially convergent.*

3 Main results

In this section, first we introduce the notions of $T-$contraction, $T-$contrative and then we extend the Banach Contraction Principle and Edelstein's fixed point Theorem given in [1] and [2].

Definition 3.1 ([1]) *Let (M,d) be a cone metric space and $T,S : M \longrightarrow M$ two functions. A mapping S is said to be a $T-$*contraction *if there is $a \in [0,1)$ constant such that*

$$d(TSx, TSy) \leq ad(Tx, Ty) \tag{3.1}$$

for all $x, y \in M$.

Example 3.2 *Let $E = (C_{[0,1]}, \mathbb{R})$, $P = \{\varphi \in E : \varphi \geq 0\} \subset E$, $M = \mathbb{R}$ and $d(x,y) = |x-y|e^t$, where $e^t \in E$. Then (M,d) is a cone metric space. We consider the functions $T,S : M \longrightarrow M$ defined by $Tx = e^{-x}$ and $Sx = 2x+1$. Then*

(i) *It is clear that S is not a contraction;*

(ii) *S is a $T-$contraction. In fact,*

$$\begin{aligned} d(TSx, TSy) &= |TSx - TSy|e^t \\ &= \frac{1}{e}|e^{-x} - e^{-y}||e^{-x} - e^{-y}|e^t \\ &\leq \frac{2}{e}|e^{-x} - e^{-y}|e^t = \frac{2}{e}d(Tx, Ty). \end{aligned}$$

The next result extend the Theorem 1 of Guang and Zhang [2], and Theorem 2.6 of Beiranvand, Moradi, Omid and Pazandeh [1].

Theorem 3.3 *Let (M,d) be a complete cone metric space, P be a normal cone with normal constant K, in addition let $T : M \longrightarrow M$ be an one to one and continuous function and $S : M \longrightarrow M$ a $T-$contraction continuous function. Then*

1. *For every $x_0 \in M$,*
$$\lim_{n\to\infty} d(TS^n x_0, TS^{n+1}x_0) = 0;$$
2. *There is $y_0 \in M$ such that*
$$\lim_{n\to\infty} TS^n x_0 = y_0;$$

3. *If T is subsequentially convergent, then $(S^n x_0)$ has a convergent subsequence;*

4. *There is a unique $z_0 \in M$ such that*

$$Sz_0 = z_0;$$

5. *If T is a sequentially convergent, then for each $x_0 \in M$ the iterate sequence $(S^n x_0)$ converges to z_0.*

Proof: For every $x_1, x_2 \in M$,

$$\begin{aligned} d(Tx_1, Tx_2) &\leq d(Tx_1, TSx_1) + d(TSx_1, TSx_2) + d(TSx_2, Tx_2) \\ &\leq d(Tx_1, TSx_1) + ad(Tx_1, Tx_2) + d(TSx_2, Tx_2) \end{aligned}$$

so,

$$d(Tx_1, Tx_2) \leq \frac{1}{1-a}\left[d(Tx_1, TSx_1) + d(TSx_2, Tx_2)\right]. \tag{3.2}$$

Now, choose $x_0 \in M$ and define the Picard iteration associated to S, (x_n) given by $x_{n+1} = Sx_n = S^n x_0$, $n = 0, 1, 2, \ldots$

$$d(Tx_n, Tx_{n+1}) = d(TS^n x_0, TS^{n+1} x_0) \leq ad(TS^{n-1} x_0, TS^n x_0)$$

hence,

$$d(TS^n x_0, TS^{n+1} x_0) \leq a^n d(Tx_0, TSx_0). \tag{3.3}$$

Since P is a normal cone with normal constant K, we get

$$\|d(TS^n x_0, TS^{n+1} x_0)\| \leq a^n K \|d(Tx_0, TSx_0)\|$$

which implies that

$$\lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = 0. \tag{3.4}$$

therefore, for $m, n \in \mathbb{N}$ with $m > n$, by (3.2) and (3.3) we have

$$\begin{aligned} d(Tx_n, Tx_m) &= d(TS^n x_0, TS^m x_0) \\ &\leq \frac{1}{1-a}\Big[d(TS^n x_0, TS^{n+1} x_0) + d(TS^{m+1} x_0, TS^m x_0)\Big] \\ &\leq \frac{1}{1-a}\Big[a^n d(Tx_0, TSx_0) + a^m d(Tx_0, TSx_0)\Big], \end{aligned}$$

hence,

$$d(TSx_0, TS^m x_0) \leq \frac{a^n + a^m}{1-a} d(Tx_0, TSx_0). \tag{3.5}$$

Taking norm to inequality above, we obtain that

$$\|d(TS^n x_0, TS^m x_0)\| \leq \frac{a^n + a^m}{1-a} K \|d(Tx_0, TSx_0)\|.$$

Consequently,

$$\lim_{n,m\to\infty} d(TS^n x_0, TS^m x_0) = 0. \tag{3.6}$$

Which prove 1. On the other hand, (3.6) implies that $(TS^n x_0)$ is a Cauchy sequence in M. By the completeness of M, there is $y_0 \in M$ such that

$$\lim_{n\to\infty} TS^n x_0 = y_0. \tag{3.7}$$

Proving in this way assertion 2. Now, if T is subsequentially convergent, then $(S^n x_0)$ has a convergent subsequence. So, there exist $z_0 \in M$ and $(n_i)_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} S^{n_i} x_0 = z_0, \tag{3.8}$$

since T is continuous we have,

$$\lim_{i\to\infty} TS^{n_i} x_0 = Tz_0 \tag{3.9}$$

from equality (3.7) we conclude that

$$Tz_0 = y_0. \tag{3.10}$$

Since S is continuous, (and also by using (3.8)) then

$$\lim_{i\to\infty} S^{n_i+1} x_0 = Sz_0$$

as well as,

$$\lim_{i\to\infty} TS^{n_i+1} x_0 = TSz_0. \tag{3.11}$$

Again by (3.7), the following equality holds,

$$\lim_{i\to\infty} TS^{n_i+1} x_0 = y_0$$

hence, $TSz_0 = y_0 = Tz_0$. Since T is injective,then $Sz_0 = z_0$, so S has a fixed point. Therefore assertion 3. is proved. On the other hand, since T is one to one and S is a $T-$contraction, S has a unique fixed point. i.e., conclusion 4.

Finally, if T is sequentially convergent, $(S^n x_0)$ is convergent to z_0, that is,

$$\lim_{n\to\infty} S^n x_0 = z_0,$$

proving in this way conclusion 5. which finishes the proof of the theorem. □

Corollary 3.4 ([2], Theorem 1) *Let (M, d) be a complete cone metric space $P \subset E$ be a normal cone with normal constant K. Suppose $S : M \longrightarrow M$ is a contraction function then S has a unique fixed point in M and for any $x_0 \in M$ $(S^n x)$ converges to the fixed point.*

Now, if we take $E = \mathbb{R}_+$ in Theorem 3.3 we obtain the following

Corollary 3.5 (Theorem 2.6, [1]) *Let (M, d) be a complete metric space and $T : M \longrightarrow M$ be an one to one, continuous and subsequentially convergent mapping. Then for every $T-$contraction continuous function $S : M \longrightarrow M$ has a unique fixed point. Moreover, if T is sequentially convergent, then for each $x_0 \in M$, the sequence $(S^n x_0)$ converge to the fixed point of S.*

If we take $E = \mathbb{R}$ and $Tx = x$ in the Theorem 3.3 then we obtain the Banach's Contraction Principle

Corollary 3.6 *Let (M, d) be a complete metric space and $S : M \longrightarrow M$ is a contraction mapping. Then S has a unique fixed point.*

The following result is the localization of the Theorem 3.3.

Theorem 3.7 *Let (M, d) be a complete cone metric space, $P \subset E$ be a normal cone with normal constant K and $T : M \longrightarrow M$ be an injective, continuous and subsequentially mapping. For $c \in E$ with $0 \ll c$, $x_0 \in M$, set*

$$B(Tx_0, c) = \{y \in M \,:\, d(Tx_0, y) \leq c\}.$$

Suppose $S : M \longrightarrow M$ is a $T-$contraction continuous mapping for all $x, y \in B(Tx_0, c)$ and $d(TSx_0, Tx_0) \leq (1 - a)c$. Then S has a unique fixed point in $B(Tx_0, c)$.

Proof: We only need to prove that $B(Tx_0, c)$ is complete and $TSx \in B(Tx_0, c)$ for all $Tx \in B(Tx_0, c)$. Suppose that (y_n) is also a Cauchy sequence in M. By the completeness of M, there exist $y \in M$ such that $y_n \longrightarrow y$, $(n \to \infty)$.

Thus, we have

$$d(Tx_0, y) \leq d(y_n, Tx_0) + d(y_n, y) \leq c + d(y_n, y)$$

since $y_n \longrightarrow y$, $(n \to \infty)$, $d(y_n, y) \longrightarrow 0$. Hence $d(Tx_0, y) \leq c$ and $y \in B(Tx_0, c)$. Therefore, $B(Tx_0, c)$ is complete.

On the other hand, for every $Tx \in B(Tx_0, c)$,

$$\begin{aligned} d(Tx_0, TSx) &\leq d(TSx_0, Tx_0) + d(TSx_0, TSx) \\ &\leq (1-a)c + ad(Tx_0, Tx) \leq (1-a)c + ac = c. \end{aligned}$$

I.e., $TSx \in B(Tx_0, c)$, and the proof is done. □

Corollary 3.8 *Let (M, d) be a complete cone metric space, $P \subset E$ be a normal cone with normal constant K and $T : M \longrightarrow M$ be an one to one, continuous and subsequentially convergent mapping. Let suppose that $S : M \longrightarrow M$ is a mapping such that, S^n is a $T-$contraction for some $n \in \mathbb{N}$ and furthermore a continuous function. Then S has a unique fixed point in M.*

Proof: From Theorem 3.3, we have that S^n has a unique fixed point $z_0 \in M$, that is, $S^n z_0 = z_0$. But $S^n(Sz) = S(S^n z) = Sz$, so $S(z)$ is also fixed point of S^n. Hence $Sz = z$, i.e., z is a fixed point of S. Since the fixed point of S is also fixed point of S^n, then the fixed point of S is unique. □

Example 3.9 *Let $E = (C_{[0,1]}, \mathbb{R})$, $P = \{\varphi \in E : \varphi \geq 0\} \subset E$,*
$M = [1, +\infty)$ and $d : M \times M \longrightarrow E$ defined by $d(x, y) = |x - y|e^t$, where $\varphi(t) = e^t \in E$. Then (M, d) is a complete cone metric space. Now we will consider the following functions, $TS : M \longrightarrow M$ defined by $Tx = 1 + \ln x$ and $Sx = 2\sqrt{x}$.

It is evident that S is not a contraction mapping, but it is a $T-$contraction because,

$$\begin{aligned} d(TSx, TSy) &= |TSx - TSy|e^t = \frac{1}{2}|\ln x - \ln y|e^t \\ &= |Tx - Ty|e^t \leq \frac{1}{2}d(Tx, Ty). \end{aligned}$$

Also, T is one to one, continuous and subsequentially convergent. Therefore, by Theorem 3.3 T has a unique fixed point, $z_0 = y$.

The following example shows that we can not omit the subsequentially convergence of the function T in the Theorem 3.3 (5).

Example 3.10 *Consider the example 3.2. Let $E = (C_{[0,1]}, \mathbb{R})$, $P = \{\varphi \in E : \varphi \geq 0\}$, $M = \mathbb{R}$ and $d : M \times M \longrightarrow E$ defined by $d(x, y) = |x - y|e^t$ where $e^t \in E$. Then (M, d) is a complete cone metric space. Let $T, S : M \longrightarrow M$ be two functions defined by $Tx = e^{-x}$ and $Sx = 2x + 1$.*

It is clear that S is a $T-$contraction, but T is not subsequentially convergent, because $Tn \to 0$, $(n \to \infty)$ but the sequence (n) has not any convergent subsequence and S has not a fixed point. □

Definition 3.11 *Let (M, d) be a cone metric space and $T, S : M \longrightarrow M$ two functions. A mapping S is said to be a $T-$*contractive *if for each $x, y \in M$ such that $Tx \neq Ty$ then*

$$d(TSx, TSy) < d(x, y).$$

It is clear that every $T-$contraction function is $T-$contractive, but the converse is not true.

Example 3.12 *1. Let $E = (C_{[0,1]}, \mathbb{R})$, $P = \{\varphi \in E : \varphi \geq 0\} \subset E$, $M = [1, +\infty)$ and $d : M \times M \longrightarrow E$ defined by $d(x, y) = |x-y|e^t$, where $e^t \in E$. Then (M, d) is a cone metric space.*

Let $T, S : M \longrightarrow M$ be two functions defined by $Tx = x$ and $Sx = \sqrt{x}$. Then:

i.- *S is a $T-$contractive function;*

ii.- *S is not a $T-$contraction mapping.*

2. Let $E = (C_{[0,1]}, \mathbb{R})$, $P = \{\varphi \in E : \varphi \geq 0\} \subset E$, $M = [0, 1/2]$ and $d : M \times M \longrightarrow E$ defined by $d(x, y) = |x - y|e^t$, where $e^t \in E$. Obviously (M, d) is a cone metric space and the function $S : M \longrightarrow M$ defined by $Sx = \dfrac{x^2}{\sqrt{2}}$ is not contractive. If $T : M \longrightarrow M$ is defined by $Tx = x^2$, then S is $T-$contractive, because:

$$\begin{aligned} d(TSx, TSy) &= |TSx - TSy|e^t = \left| \frac{x^4}{2} - \frac{y^4}{2} \right| e^t = \frac{1}{2}|x^2 + y^2||Tx - Ty|e^t \\ &< |Tx - Ty|e^t = d(Tx, Ty). \end{aligned}$$

□

The following result extend the Theorem 2 of [1] and Theorem 2.9 of [2].

Theorem 3.13 *Let (M, d) be a compact cone metric space, P be a normal cone with normal constant K and $T, S : M \longrightarrow M$ functions such that T is injective, continuous and S is $T-$contractive mapping. Then,*

i.- *S has a unique fixed point;*

ii.- *For any $x_0 \in M$ the sequence iterates $(S^n x_0)$ converges to the fixed point of S.*

Proof: In first we are going to show that S is a continuous function. Let $\lim\limits_{n\to\infty} x_n = x$, we want to prove that $\lim\limits_{n\to\infty} Sx_n = Sx$. Since S is $T-$contractive, we get

$$d(TSx_n, TSx) \leq d(Tx_n, Tx)$$

so,

$$\|d(TSx_n, TSx)\| \leq K\|d(Tx_n, Tx)\|.$$

Now, since T is continuous, we have

$$\lim_{n\to\infty} \|d(TSx_n, TSx)\| = 0$$

also that,

$$\lim_{n\to\infty} d(TSx_n, TSx) = 0$$

therefore,

$$\lim_{n\to\infty} TSx_n = TSx. \tag{3.12}$$

Let (Sx_{n_i}) be an arbitrary convergent subsequence of (x_n). There is a $y \in M$ such that

$$\lim_{i\to\infty} Sx_{n_i} = t.$$

By the continuity of T we infer,

$$\lim_{i\to\infty} TSx_{n_i} = Ty. \tag{3.13}$$

By (3.12) and (3.13) we conclude that $TSx = Ty$. Since T is one to one then, $Sx = y$. Hence, every convergence subsequence of (Sx_n) converge to Sx. From the fact M a compact cone metric space, we arrive to the conclusion that S is a continuous function.

Now, because of T and S are continuous functions, then the function $\varphi : M \longrightarrow P$ defined by $\varphi(y) = d(TSy, Ty)$, for all $y \in M$, is continuous on M and from the compactness of M, the function φ attains its minimum, say at $x \in M$.

If $Sx \neq x$, then

$$\varphi(Sx) = d(TS^2x, TSx) < d(TSx, Tx) = \varphi(x)$$

which is a contradiction, So $Sx = x$ proving in this form part i. Choose $x_0 \in M$ and set $a_n = d(TS^n x_0, Tx)$. Since

$$a_{n+1} = d(TS^{n+1}x_0, Tx) = d(TS^{n+1}x_0, TSx) \leq d(TS^n x_0, Tx) = a_n,$$

then (a_n) is a non increasing sequence of non negative real numbers and so it has a limit, say a, that is

$$a = \lim_{n\to\infty} a_n \quad \text{or} \quad \lim_{n\to\infty} d(TS^n x_0, Tx) = a.$$

By compactness, $(TS^n x_0)$ has a convergent subsequence $(TS^{n_i}x_0)$ i.e.,

$$\lim_{i\to\infty} TS^{n_i}x_0 = z, \tag{3.14}$$

from the sequentially convergence of T, there exists $w \in M$ such that

$$\lim_{i\to\infty} S^{n_i}x_0 = w$$

so,

$$\lim_{i\to\infty} TS^{n_i}x_0 = Tw. \tag{3.15}$$

By (3.14) and (3.15), $Tw = z$. Then $d(Tw, Tx) = a$. Now we are going to show that $Sw = x$. If $Sw \neq x$, then

$$\begin{aligned} a &= \lim_{n\to\infty} d(TS^n x_0, Tx) = \lim_{i\to\infty} d(TS^{n_i}x_0, Tx) = d(TSw, Tx) \\ &= d(TSw, TSx) < d(Tw, Tx) = a \end{aligned}$$

which is a contradiction. In this way, we get that $Sw = x$ and hence,

$$a = \lim_{i\to\infty} d(TS^{n_i+1}x_0, Tx) = d(TSw, Tx) = 0$$

Therefore, $\lim\limits_{n\to\infty} TS^n x_0 = Tx$. Finally condition T sequentially convergent implies $\lim\limits_{n\to\infty} S^n x_0 = x$, which finalize the proof. $\square$

If we take $E = \mathbb{R}$ and $Tx = x$ in Theorem 3.13, we obtain the Edelstein's fixed point theorem (see, e.g., [3]).

Example 3.14 *We must recall example 3.12 (2). Let $E = (C_{[0,1]}, \mathbb{R})$, $P = \{\varphi \in E : \varphi \geq 0\} \subset E$, $M = [0,1]$ and $d(x,y) = |x-y|e^t$, $e^t \in E$. It is clear that M is a compact cone metric space. The functions $T, S : M \longrightarrow M$ defined by $Tx = x^2$ and $Sx = \dfrac{x^2}{\sqrt{2}}$ satisfy that T is injective and continuous whereas S is $T-$contractive. So by Theorem 3.13 we have that S has a unique fixed point, $x = 0$.* $\square$

References

[1] A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh, *Two fixed point theorem for special mapping*, arXiv: 0903.1504v1.

[2] Huang Long - Guang, Zhan Xian, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. 332 (2007), 1468 - 1476.

[3] Kazimierz Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, NY, 1990.

[4] G. Jungck, S. Radenovic, S. Radojevic and V. Rakocevic, *Common fixed point theorems for weakly compatible pairs on cone metric spaces*, to appear.

[5] Dejan Ilic and Vladimir Rakocevic, *Common fixed point for maps on cone metric space*, J. Math. Anal. Appl. 341, **(2)**, (2008), 876 - 882.

[6] Mujahid Abbas and B. E. Rhoades, *Fixed and Periodic results in cone metric spaces*, Appl. Math. Lett. (2008).

[7] P. Raja and S. M. Vaezpour, *Some extensions of Banach's contraction principle in complete cone metric spaces, fixed point theory and applications*, (2008), 11 p.

[8] Dariusz Wardowski, *End points and fixed points of set - valued contractions in cone metric spaces nonlinear Analysis*, 2009.

JOSÉ R. MORALES
Departamento de Matemáticas, Facultad de Ciencias,
Universidad de Los Andes
Mérida 5101, Venezuela
e-mail: moralesj@ula.ve

EDIXÓN ROJAS
Departamento de Matemáticas, Facultad de Ciencias,
Universidad de Los Andes
Mérida 5101, Venezuela
e-mail: edixonr@ula.ve