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On the complexity of the family of compact subsets of $\mathbb{Q}$

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Resumen

Mostramos que $K(\mathbb{Q})$, la familia de subconjuntos compactos de $\mathbb{Q}$, es Π^1_1-completa en el cubo de Cantor $2^{\mathbb{Q}}$.

Palabras claves: Conjuntos coanalíticos completos, Teorema de Hurewicz, Teoría descriptiva de conjuntos.

Abstract

We show that $K(\mathbb{Q})$, the collection of compact subsets of $\mathbb{Q}$, is a Π^1_1-complete subset of the Cantor cube $2^{\mathbb{Q}}$.

key words. Complete coanalytic sets, Hurewicz's theorem, descriptive set theory.

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1 Introduction and preliminaries

In this note we will show that $K(\mathbb{Q})$, the collection of compact subsets of the rationals, is a complete coanalytic subset of cantor cube $2^{\mathbb{Q}}$. This result is a variation of a classical theorem of Hurewicz saying that $K(\mathbb{Q})$ is a complete coanalytic subset of $K(\mathbb{R})$ with the Vietoris topology (see [1, 3]). The more general problem, where instead of $\mathbb{Q}$ we consider any countable topological space, was studied in [2].

Since both $K(\mathbb{R})$ and $2^{\mathbb{Q}}$ are perfect Polish spaces, it is a classical result that they are Borel isomorphic, so there is $f : K(\mathbb{R}) \to 2^{\mathbb{Q}}$ a Borel isomorphism [1, Theorem 15.6]. Our result would be a trivial consequence of this fact if one can find such f that leaves $K(\mathbb{Q})$ invariant, that is to say, such that $L \in K(\mathbb{Q})$ iff $f(L) \in K(\mathbb{Q})$. Since, in that case, by Hurewicz's theorem $K(\mathbb{Q})$ is a complete coanalytic subset of $K(\mathbb{R})$ and therefore so is the image of $K(\mathbb{Q})$ under f. We do not know if such f exists.

Now we fix some terminology and state some basic facts.

Let X be a Polish space and $A \subseteq X$. We say that A is **Π^1_1-complete** if A is Π^1_1 and for all Polish space Y and all Π^1_1 subset $B \subseteq Y$ there is Borel function $f : Y \to X$ such that $f^{-1}(A) = B$.

Let X and Y be Polish spaces and $A \subseteq X$ and $B \subseteq Y$. We say that A is **Borel reducible** to B if there is a Borel function $f : X \to Y$ such that $f^{-1}(B) = A$. The following proposition is easy to show.

Proposition 1.1 *Let X, Y be polish spaces and $A \subseteq X$ and $B \subseteq Y$. If B is Π^1_1, A is Π^1_1-complete and A is Borel reducible to B, then B is Π^1_1-complete.*

Let $\omega^{<\omega}$ be the collection of all finite sequence of natural number. We denote by $s \prec t$ if t extends s, clearly $\prec$ is a partial order on $\omega^{<\omega}$. We denote by $s\widehat{\ }t$ the concatenation of the sequences s and t. By $s\widehat{\ }k$ we denote the sequence $s\widehat{\ }\langle k\rangle$. A tree over $\mathbb{N}$ is a collection of finite sequences closed under initial segments, i.e., if $s \in T$ and $t \prec s$, then $t \in T$. Let $Tree$ be the collection of all trees over $\mathbb{N}$. The body of a tree T is the collection $[T]$ of all $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha \restriction n \in T$ for all $n \in \mathbb{N}$. Such α is called an infinite branch of T. A tree is well founded if $[T] = \emptyset$. Let WF be the collection of all well founded trees over $\mathbb{N}$. We regards trees as elements of the polish space $2^{\omega^{<\omega}}$. Then $Tree$ is a closed subset of $2^{\omega^{<\omega}}$. Moreover, WF is the prototypical Π^1_1-complete set [1, Theorem 27.1]. We will use a variation of this collection. Let $Tree_2$ be the collection of all binary trees. We say that $\alpha \in 2^{\mathbb{N}}$ has infinite many ones, if $\{n \in \mathbb{N} : \alpha(n) = 1\}$ is infinite. Let N be the set of all $\alpha \in 2^{\mathbb{N}}$ with infinite many ones. Let WF_2 be the collection of all binary trees T such that $[T] \cap N = \emptyset$. Notice that, by Konig lemma, $T \in Tree_2$ is infinite iff $[T] \neq \emptyset$. However, as we show in proposition 2.1, WF_2 is Π^1_1-complete.

The order of Kleene-Brouwer over $\omega^{<\omega}$, denoted by $\prec_{KB}$, is defined as follows: Let $s = (s_0, s_1, \dots, s_{n-1})$ and $t = (t_0, t_1, \dots, t_{m-1})$ in $\omega^{<\omega}$. Then $s \prec_{KB} t$, if

(i) $t \prec s$ (s extends t), or

(ii) There is $i < \min\{m, n\}$ such that $s_j = t_j$ for all $j < i$ and $s_i < t_i$

We put $s \preceq_{KB} t$ if $s \prec_{KB} t$ o $s = t$. An interesting fact about $\prec_{KB}$ is that a tree T over $\mathbb{N}$ is well founded iff $\prec_{KB}$ is a well order over T (see [1, 3]). The interval determined by two sequences s, t is denoted by $(s, t)_{KB}$. The order topology associated to $\prec_{KB}$ will be denoted by τ_{KB}.

It is well known that every countable metric space without isolated points is homeomorphic to the rationals (see for instance [1, pag. 40]). In particular, this is the case of $(\omega^{<\omega}, \tau_{KB})$, and thus will work with the space $(\omega^{<\omega}, \tau_{KB})$ instead of $\mathbb{Q}$. We state this result for later reference.

Lemma 1.2 *$(\omega^{<\omega}, \tau_{KB})$ is homeomorphic to $\mathbb{Q}$.*

2 $K(\mathbb{Q})$ is Π^1_1-complete in $2^{\mathbb{Q}}$

The following result is known (see [1, Exercise 27.3]) but we include its proof for the sake of completeness.

Proposition 2.1 *WF_2 is Π^1_1-complete in $2^{2^{<\mathbb{N}}}$.*

Proof: Let us first check that WF_2 is Π^1_1.

It is well known that the collection of well founded trees WF is Π^1_1-complete [1, Theorem 27.1]. By proposition 1.1, it suffices to show that WF is Borel reducible to WF_2.

We denote $00\cdots 0$, k times, by 0^k. Let us consider the function $\varphi : \mathbb{N}^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ given by

$$\varphi(n_0, n_1, \dots, n_m) = 0^{n_0} 1 0^{n_1} 1 \cdots 0^{n_m}.$$

and let $\Phi : Tree \to Tree_2$ given by

$$\Phi(T) = \{s \in 2^{<\mathbb{N}} :\ (\exists t \in T)(\, s \preceq \varphi(t))\}$$

Then Φ is a Borel map. Let us check that $\Phi^{-1}(WF_2) = WF$.

(i) Suppose T is not a well founded tree and let $\alpha \in [T]$. Then $\varphi(\alpha \restriction n) \in \Phi(T)$ for all $n \in \mathbb{N}$ and $\varphi(\alpha \restriction n)$ has n ones. Thus $\varphi(\alpha)$ is a branch of $\Phi(T)$ with infinite many ones.

(ii) Suppose now that $\Phi(T) \notin WF_2$ and let β be a branch of $\Phi(T)$ with infinite many ones. Suppose $s\widehat{\ }1 \prec \beta$, then $s = \varphi(t)$ for some $t \in T$. From this it follows easily that T has an infinite branch.

□

Lemma 2.2 *(i) $\inf\{s\widehat{\ }1\widehat{\ }0^n\widehat{\ }t_n :\ n \in \mathbb{N}\} = s\widehat{\ }0$ for any $s, t_n \in 2^{<\mathbb{N}}$.*

(ii) If $\alpha \in 2^{\mathbb{N}}$ has infinite many ones, then $\{\alpha \restriction n :\ n \in \mathbb{N}\}$ has no infimun.

Let $(n_k)_k$ be an increasing sequence of integer and $t_k \in 2^{<\mathbb{N}}$. Then $0^{n_k}\widehat{\ }t_k$ has no infimun.

(iii) If $T \in WF_2$, then there is no a strictly $\prec_{KB}$-increasing sequence in T.

(iv) Let $T \in WF_2$ and $(s_i)_i$ a strictly $\prec_{KB}$-decreasing sequence in T. Then $(1^\frown s_i)_i$ either converges to $\langle 0 \rangle$ or to a sequence of the form $1^\frown t^\frown 0$ for some $t \in T$.

Proof: (i) and (ii) are easy and left to the reader. To see (iii) let (s_i) be a strictly $\prec_{KB}$-increasing sequence of elements of T. We can assume that the length of s_i is strictly increasing. By passing to a subsequence if necessary, from the definition of $\prec_{KB}$ we get sequences u_i, v_i such that $s_i = u_i{}^\frown 1^\frown v_i$, $u_{i+1}{}^\frown 0 \prec s_i$ and $u_i{}^\frown 1 \prec u_{i+1}$. From this it follows that $\bigcup_i u_i{}^\frown 1$ is a branch of $[T]$ with infinite many ones.

(iv) Suppose $(s_i)_i$ is a strictly $\prec_{KB}$-decreasing sequence in T with $T \in WF_2$. Then it follows from the definition of $\prec_{KB}$ that there is a subsequence $(s_{i_k})_k$ such that one of the following holds:

(a) $s_{i_{k+1}} \prec s_{i_k}$ for all k and therefore $\alpha = \bigcup_k s_{i_k}$ is eventually equal to cero. If α is equal to cero, then $(1^\frown s_i)_i$ converges to $\langle 0 \rangle$. Otherwise, there is $t \in T$ such that $t^\frown 1^\frown 0^n \prec \alpha$ for all n and thus $(1^\frown s_i)_i$ converges to $1^\frown t^\frown 0$.

(b) there are $t, u_k \in 2^{<\mathbb{N}}$ such that $s_{i_k} = t^\frown 1^\frown 0^{n_k}{}^\frown u_k$ where $(n_k)_k$ is strictly increasing. In this case, it follows from part (i) that $(s_i)_i$ converges to $t^\frown 0$ for some $t \in T$.

□

Theorem 2.3 *$K(\mathbb{Q})$ is Π^1_1-complete in $2^{\mathbb{Q}}$*

Proof: Let us first check that $K(\mathbb{Q})$ is coanalytic. By lemma 1.2, we can work in $(\omega^{<\omega}, \tau_{KB})$.

Let $\psi : Tree_2 \to 2^{2^{<\mathbb{N}}}$ given by

$$\psi(T) = \{1^\frown s : \; s \in T\} \cup \{1^\frown s^\frown 0 : \; s^\frown 1 \in T\} \cup \{\langle 0 \rangle\}.$$

We will show that ψ is a continuous reduction from WF_2 into $K(\mathbb{Q})$. That is to say, $T \in WF_2$ iff $\psi(T)$ is compact as a subset of $(2^{<\mathbb{N}}, \tau_{KB})$.

It is easy to check that ψ is continuous. Suppose $T \notin WF_2$ and let $\alpha \in [T]$ be a sequence with infinite many ones. Then, from lemma 2.2 we know that $\{1^\frown \langle \alpha \restriction n \rangle : \; n \in \mathbb{N}\}$ is a decreasing sequence in $\psi(T)$ without infimun.

Conversely, suppose $T \in WF_2$ and let $(t_i)_i$ be a strictly $\prec_{KB}$-monotone sequence in $\psi(T)$. We will show that $(t_i)_i$ converges in $\psi(T)$. There are two cases to consider.

(i) Suppose $t_i = 1^\frown s_i$ with $s_i \in T$ for all i. Then, $(s_i)_i$ is also strictly $\prec_{KB}$-monotone. Since

$T \in WF_2$, by lemma 2.2 we conclude that $(s_i)_i$ is strictly $\prec_{KB}$-decreasing and moreover $(t_i)_i$ converges either to $\langle 0 \rangle$ or converges to $1^\frown t^\frown 0$ for some $t \in T$.

(ii) Suppose $t_i = 1^\frown s_i{}^\frown 0$ with $s_i{}^\frown 1 \in T$ for all i. Moreover, we can assume that $s_i{}^\frown 0 \notin T$ for all i, otherwise we are in case (i). It is clear that $(s_i{}^\frown 0)_i$ is strictly $\prec_{KB}$-monotone. Since $s_i \in T$ and $s_i{}^\frown 0 \notin T$, then $s_i{}^\frown 0$ and $s_{i+1}{}^\frown 0$ are not $\prec$-comparable. From this it follows that $(s_i)_i$ is strictly $\prec_{KB}$-monotone. Therefore $(s_i)_i$ converges to a sequence $t^\frown 0$ for some $t \in T$. Then it is easy to check that t_i converges to $1^\frown t^\frown 0$.

Thus by proposition 2.1 and theorem 1.1 we conclude that $K(\mathbb{Q})$ is Π^1_1-complete.

□

Notice that the tree $\{0^n : n \in \mathbb{N}\}$ is not τ_{KB}-compact, as it has no accumulation point. This fact and the following lemma explain why we have defined the function ψ as we did in the proof of the previous theorem

Lemma 2.4 *Let T be a binary tree. Then*

$$\alpha(T) = T \cup \{s^\frown 0 : s^\frown 1 \in T\}$$

is τ_{KB}-closed. In particular, $2^{<\mathbb{N}}$ is τ_{KB}-closed.

Proof: Let $O = \omega^{<\omega} \setminus \alpha(T)$. We will show that O is open. Let $u \in O$. In particular, $u \neq \emptyset$ and $u \notin T$.

Case 1: Suppose $u = v^\frown 0$. Thus $v^\frown 1 \notin T$. Then $u \in (v^\frown 0^\frown 0, v^\frown 1)_{KB} \subseteq O$.

Case 2: Suppose $u = v^\frown k$ for some $k \geq 1$. Thus $u \notin T$. Then $u \in (v^\frown k^\frown 0, v)_{KB} \subseteq O$.

□

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