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Mannheim curves in terms of its timelike biharmonic partner curves in the Lorentzian Heisenberg Group Heis^3

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Abstract

In this paper, we study Mannheim curves in the Lorentzian Heisenberg group Heis^3. We characterize Mannheim curves in terms of its horizontal biharmonic partner curves in the Lorentzian Heisenberg group Heis^3.

key words. Heisenberg group, biharmonic curve, Mannheim curves.

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1 Introduction

In modern physics (especially general relativity), spacetime is represented by a Lorentzian Manifold. Minkowski spacetime is a simple example of a Lorentzian Manifold. Lorentzian geometry plays an important role in that transition between modern differential geometry and mathematical physics.

The notion of Mannheim curves was discovered by A. Mannheim in 1878. These curves in Euclidean 3-space are characterized in terms of the curvature and torsion as follows: A space curve is a Mannheim curve if and only if its curvature and torsion satisfy the relation

$$\kappa(s) = \lambda\left(\kappa^2(s) + \tau^2(s)\right)$$

for some constant λ.

Harmonic maps $f : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds are the critical points of the energy

$$E(f) = \frac{1}{2}\int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f) = \mathrm{trace}\nabla df. \tag{1.2}$$

As suggested by Eells and Sampson in [8], we can define the bienergy of a map f by

$$E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [10], showing that the Euler–Lagrange equation associated to E_2 is

$$\tau_2(f) = -\mathcal{J}^f(\tau(f)) = -\Delta\tau(f) - \mathrm{trace} R^N(df, \tau(f))\, df = 0, \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of f . The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper, we study Mannheim curves in the Lorentzian Heisenberg group Heis^3. We characterize Mannheim curves in terms of its biharmonic partner curves in the Lorentzian Heisenberg group Heis^3.

2 The Lorentzian Heisenberg Group Heis^3

The Lorentzian Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multiplication:

$$(\overline{x}, \overline{y}, \overline{z})(x, y, z) = (\overline{x} + x, \overline{y} + y, \overline{z} + z - \overline{x}y + x\overline{y}).$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Lorentz metric g is given by

$$g = -dx^2 + dy^2 + (xdy + dz)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1 = \frac{\partial}{\partial z}, \quad \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad \mathbf{e}_3 = \frac{\partial}{\partial x} \tag{2.1}$$

for which we have the Lie products

$$[\mathbf{e}_2, \mathbf{e}_3] = 2\mathbf{e}_1, \quad [\mathbf{e}_3, \mathbf{e}_1] = 0, \quad [\mathbf{e}_2, \mathbf{e}_1] = 0$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, \quad g(\mathbf{e}_3, \mathbf{e}_3) = -1.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above, the following is true:*

$$\nabla = \begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2.2}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = -g(R(X,Y)Z, W).$$

Moreover we put

$$R_{abc} = R(\mathbf{e}_a, \mathbf{e}_b)\mathbf{e}_c, \;\; R_{abcd} = R(\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c, \mathbf{e}_d),$$

where the indices a, b, c and d take the values $1, 2$ and 3.

$$\begin{aligned} R_{232} &= -3R_{131} = -3\mathbf{e}_3, \\ R_{133} &= -\; R_{122} = \mathbf{e}_1, \\ R_{233} &= -3R_{121} = -3\mathbf{e}_2, \end{aligned}$$

and

$$R_{1212} = -1, \quad R_{1313} = 1, \quad R_{2323} = -3. \tag{2.3}$$

3 Timelike Biharmonic Curves In The Lorentzian Heisenberg Group Heis^3

Let $\gamma : I \longrightarrow Heis^3$ be a timelike curve on the Lorentzian Heisenberg group Heis^3 parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group Heis^3 along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\nabla_{\mathbf{T}}\mathbf{T} = \kappa\mathbf{N},$$
$$\nabla_{\mathbf{T}}\mathbf{N} = \kappa\mathbf{T} + \tau\mathbf{B}, \quad (3.1)$$
$$\nabla_{\mathbf{T}}\mathbf{B} = -\tau\mathbf{N},$$

where κ is the curvature of γ and τ is its torsion. With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T} \times \mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned}$$

Theorem 3.1. *(see [18]) Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic timelike curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. γ is a timelike non-geodesic biharmonic curve if and only if*

$$\kappa = \text{constant} \neq 0,$$
$$\kappa^2 - \tau^2 = -1 + 4B_1^2, \quad (3.2)$$
$$\tau' = -2N_1B_1.$$

Corollary 3.2. *(see [18]) Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic timelike curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. γ is biharmonic if and only if*

$$\kappa = \text{constant} \neq 0,$$
$$\tau = \text{constant}, \quad (3.3)$$
$$N_1B_1 = 0,$$
$$\kappa^2 - \tau^2 = -1 + 4B_1^2.$$

Theorem 3.3. *(see [18]) Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic timelike curve on Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. If $N_1 \neq 0$ then γ is not biharmonic.*

Theorem 3.4. *(see [18]) Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic timelike biharmonic curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. If $N_1 = 0$, then*

$$\mathbf{T}(s) = \sinh\phi_0\mathbf{e}_1 + \cosh\phi_0\sinh\psi(s)\mathbf{e}_2 + \cosh\phi_0\cosh\psi(s)\mathbf{e}_3, \quad (3.4)$$

where $\phi_0 \in \mathbb{R}$.

4 Mannheim Curves In The Lorentzian Heisenberg Group Heis^3

Definition 4.1. *Let $\gamma, \beta : I \longrightarrow Heis^3$ be a unit speed non-geodesic curve. If there exists a corresponding relationship between the space curves γ and β such that, at the corresponding points of the curves, the principal normal lines of β coincides with the binormal lines of β, then β is called a Mannheim curve, and γ a Mannheim partner curve of β. The pair $\{\gamma, \beta\}$ is said to be a Mannheim pair.*

Theorem 4.2. *Let $\beta : I \longrightarrow Heis^3$ be a Mannheim curve and γ its horizontal biharmonic partner curve. Then, the parametric equation of Mannheim curve β in terms of its horizontal biharmonic partner curve γ of β are given below (4.1)*

$$\begin{aligned}
x_\beta(s) &= \frac{\lambda}{\kappa}(\cosh\phi_0\sinh[\Re s+\rho](\Re\cosh\phi_0\cosh[\Re s+\rho]+2\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho]) \\
&\quad .(\frac{\kappa}{\Re}\cosh\phi_0\sinh[\Re s+\rho]+\frac{2\kappa}{\Re^2}\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho]+\rho_1 s+\rho_2) \\
&\quad -\frac{\lambda}{\kappa}(\cosh\phi_0\cosh[\Re s+\rho]-\left(\frac{1}{\Re}\cosh\phi_0\sinh[\Re s+\rho]+\rho_0\right)\cosh\phi_0\sinh[\Re s+\rho]) \\
&\quad .(\Re\cosh\phi_0\cosh[\Re s+\rho]+2\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho]) \\
&\quad +\frac{1}{\Re}\cosh\phi_0\sinh[\Re s+\rho]+\rho_0, \\
y_\beta(s) &= \frac{\lambda}{\kappa}(\Re\cosh\phi_0\sinh[\Re s+\rho]+2\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho]) \\
&\quad .(\cosh\phi_0\cosh[\Re s+\rho]-\left(\frac{1}{\Re}\cosh\phi_0\sinh[\Re s+\rho]+\rho_0\right)\cosh\phi_0\sinh[\Re s+\rho]) \\
&\quad -\frac{\lambda}{\kappa}\cosh\phi_0\cosh[\Re s+\rho](\Re\cosh\phi_0\cosh[\Re s+\rho]+2\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho]) \\
&\quad .(\frac{\kappa}{\Re}\cosh\phi_0\sinh[\Re s+\rho]+\frac{2\kappa}{\Re^2}\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho]+\rho_1 s+\rho_2) \\
&\quad +\frac{1}{\Re}\cosh\phi_0\cosh[\Re s+\rho]+\rho_4, \\
z_\beta(s) &= \frac{\lambda}{\kappa}\cosh\phi_0\cosh[\Re s+\rho](\Re\cosh\phi_0\cosh[\Re s+\rho]+2\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho]) \\
&\quad -\frac{\lambda}{\kappa}\cosh\phi_0\sinh[\Re s+\rho](\Re\cosh\phi_0\sinh[\Re s+\rho]+2\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho]) \\
&\quad +\frac{1}{\Re}\cosh\phi_0\sinh[\Re s+\rho]-\frac{1}{\Re}\cosh^2\phi_0(-\frac{s}{2}+\frac{\sinh 2[\Re s+\rho]}{4\Re}) \\
&\quad -\frac{\rho_0}{\Re}\cosh\phi_0\cosh[\Re s+\rho]+\rho_5,
\end{aligned}$$

where ρ, ρ_0, ρ_1, ρ_2, ρ_4, ρ_5 are constants of integration and $\Re = (\pm\frac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0)$.

Proof. The covariant derivative of the vector field $\mathbf{T}$ is:

$$\nabla_{\mathbf{T}}\mathbf{T} = T_1'\mathbf{e}_1 + (T_2' + 2T_1T_3)\mathbf{e}_2 + (T_3' + 2T_1T_2)\mathbf{e}_3. \tag{4.2}$$

From (3.4), we have

$$\nabla_{\mathbf{T}}\mathbf{T} = (\psi' \cosh\phi_0 \cosh\psi(s) + 2\sinh\phi_0 \cosh\phi_0 \cosh\psi(s))\mathbf{e}_2 \tag{4.3}$$

$$+(\psi' \cosh\phi \sinh\psi(s) + 2\sinh\phi_0 \cosh\phi_0 \cosh\psi(s))\mathbf{e}_3.$$

Since $|\nabla_{\mathbf{T}}\mathbf{T}| = \kappa$ we obtain

$$\psi(s) = (\pm\frac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0)s + \rho, \tag{4.4}$$

where $\rho \in \mathbb{R}$.

Thus (3.4) and (4.4), imply

$$\mathbf{T} = \sinh\phi_0 e_1 + \cosh\phi_0 \sinh[\Re s + \rho]\, e_2 + \cosh\phi_0 \cosh[\Re s + \rho]\, e_3, \tag{4.5}$$

where $\Re = (\pm\frac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0)$.

Using (3.1) in (4.5), we obtain

$$\begin{aligned}\mathbf{T} \;\; = \;\; & (\cosh\phi_0 \cosh[\Re s + \rho], \cosh\phi_0 \sinh[\Re s + \rho], \cosh\phi_0 \cosh[\Re s + \rho]) \\ & -x(s)\cosh\phi_0 \sinh[\Re s + \rho]).\end{aligned}$$

From (2.1), we get

$$\begin{aligned}\mathbf{T} \;\; = \;\; & (\cosh\phi_0 \cosh[\Re s + \rho], \cosh\phi_0 \sinh[\Re s + \rho], \cosh\phi_0 \cosh[\Re s + \rho]) \\ & -\left(\frac{1}{\Re}\cosh\phi_0 \sinh[\Re s + \rho] + \rho_0\right)\cosh\phi_0 \sinh[\Re s + \rho]),\end{aligned}$$

where ρ_0 is constant of integration.

On the other hand, suppose that $\beta(s)$ is a Mannheim curve. Then by the definition we can assume that

$$\beta(s) = \gamma(s) + \lambda\mathbf{B}(s). \tag{4.6}$$

From (3.1) and (4.5), we get

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} \;\; = \;\; & (\Re\cosh\phi_0 \cosh[\Re s + \rho] + 2\sinh\phi_0 \cosh\phi_0 \cosh[\Re s + \rho])\mathbf{e}_2 \\ & +(\Re\cosh\phi_0 \sinh[\Re s + \rho] + 2\sinh\phi_0 \cosh\phi_0 \cosh[\Re s + \rho])\mathbf{e}_3.\end{aligned}$$

where $\Re = (\pm\frac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0)$.

By the use of Frenet formulas, we get

$$\mathbf{N} = \frac{1}{\kappa}\nabla_{\mathbf{T}}\mathbf{T}$$

$$= \frac{1}{\kappa}[(\Re\cosh\phi_0 \cosh[\Re s + \rho] + 2\sinh\phi_0 \cosh\phi_0 \cosh[\Re s + \rho])\mathbf{e}_2 \tag{4.7}$$

$$+(\Re\cosh\phi_0\sinh[\Re s+\rho]+2\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho])\mathbf{e}_3].$$

Substituting (2.1) in (4.7), we have

$$\begin{aligned}
\mathbf{N} &= \frac{1}{\kappa}((\Re\cosh\phi_0\sinh[\Re s+\rho]+2\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho]),\\
&(\Re\cosh\phi_0\cosh[\Re s+\rho]+2\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho]),\\
&(\Re\cosh\phi_0\cosh[\Re s+\rho]+2\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho])\\
&.(\frac{\kappa}{\Re}\cosh\phi_0\sinh[\Re s+\rho]+\frac{2\kappa}{\Re^2}\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho]+\rho_1 s+\rho_2)).
\end{aligned}$$

Noting that $\mathbf{T}\times\mathbf{N}=\mathbf{B}$, we have the following (4.8)

$$\begin{aligned}
\mathbf{B} &= \frac{1}{\kappa}(\cosh\phi_0\sinh[\Re s+\rho](\Re\cosh\phi_0\cosh[\Re s+\rho]+2\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho])\\
&.(\frac{\kappa}{\Re}\cosh\phi_0\sinh[\Re s+\rho]+\frac{2\kappa}{\Re^2}\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho]+\rho_1 s+\rho_2)\\
&-(\cosh\phi_0\cosh[\Re s+\rho]-\left(\frac{1}{\Re}\cosh\phi_0\sinh[\Re s+\rho]+\rho_0\right)\cosh\phi_0\sinh[\Re s+\rho])\\
&.(\Re\cosh\phi_0\cosh[\Re s+\rho]+2\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho]),\\
&(\Re\cosh\phi_0\sinh[\Re s+\rho]+2\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho])\\
&.(\cosh\phi_0\cosh[\Re s+\rho]-\left(\frac{1}{\Re}\cosh\phi_0\sinh[\Re s+\rho]+\rho_0\right)\cosh\phi_0\sinh[\Re s+\rho])\\
&-\cosh\phi_0\cosh[\Re s+\rho](\Re\cosh\phi_0\cosh[\Re s+\rho]+2\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho])\\
&.(\frac{\kappa}{\Re}\cosh\phi_0\sinh[\Re s+\rho]+\frac{2\kappa}{\Re^2}\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho]+\rho_1 s+\rho_2),\\
&\cosh\phi_0\cosh[\Re s+\rho](\Re\cosh\phi_0\cosh[\Re s+\rho]+2\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho])\\
&-\cosh\phi_0\sinh[\Re s+\rho](\Re\cosh\phi_0\sinh[\Re s+\rho]+2\sinh\phi_0\cosh\phi_0\cosh[\Re s+\rho])).
\end{aligned}$$

Next, we substitute (4.5) and (4.8) into (4.6), we get (4.1). The proof is completed.

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