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Evolute curves of biharmonic curves in the special three-dimensional ϕ-Ricci symmetric Para-Sasakian manifold $\mathbb{P}$

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Abstract

In this paper, we study evolute curve of biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. We characterize evolute curve of biharmonic curve in terms of curvature and torsion of biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Finally, we find out explicit parametric equations of evolute curve of biharmonic curve.

key words. Evolute curve, biharmonic curve, para-Sasakian manifold, curvature, torsion. **AMS subject classifications.** 53C41, 53A10.

1 Introduction

In a different setting, Chen [5] defined biharmonic submanifolds $M \subset \mathbb{E}^n$ of the Euclidean space as those with harmonic mean curvature vector field, that is $\Delta H = 0$; where is the rough Laplacian, and stated the following

Conjecture: Any biharmonic submanifold of the Euclidean space is harmonic, that is minimal.

If the definition of biharmonic maps is applied to Riemannian immersions into Euclidean space, the notion of Chen's biharmonic submanifold is obtained, so the two definitions agree.

The non-existence theorems for the case of non-positive sectional curvature codomains, as well as the

Generalized Chen's conjecture: Biharmonic submanifolds of a manifold N with $Riem^N \leq 0$ are minimal, encouraged the study of proper biharmonic submanifolds, that is submanifolds such that the inclusion map is a biharmonic map, in spheres or another non-negatively curved spaces [1, 2, 3, 5, 10].

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_{\phi}\mathcal{T}(\phi) + \mathrm{tr}R(\mathcal{T}(\phi), d\phi) \, d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study evolute curve of biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. We characterize evolute curve of biharmonic curve in terms of curvature and torsion of biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Finally, we find out explicit parametric equations of evolute curve of biharmonic curve.

2 Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

An n-dimensional differentiable manifold M is said to admit an almost para-contact Riemannian structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\phi\xi = 0, \quad \eta(\xi) = 1, g(X, \xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = X - \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for any vector fields X, Y on M [1].

Definition 2.1. A para-Sasakian manifold M is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X, Y) Z) = 0,$$

for all vector fields X, Y, Z, W orthogonal to ξ [1].

Definition 2.2. A para-Sasakian manifold M is said to be ϕ-symmetric if

$$\phi^2\left(\left(\nabla_W R\right)(X,Y)\,Z\right)=0,$$

for all vector fields X,Y,Z,W on M.

Definition 2.3. A para-Sasakian manifold M is said to be ϕ-Ricci symmetric if the Ricci operator satisfies

$$\phi^2\left(\left(\nabla_X Q\right)(Y)\right)=0,$$

for all vector fields X and Y on M and $S(X,Y)=g(QX,Y)$.

If X,Y are orthogonal to ξ, then the manifold is said to be locally ϕ-Ricci symmetric.

We consider the three-dimensional manifold

$$\mathbb{P}=\left\{\left(x^1,x^2,x^3\right)\in\mathbb{R}^3:\left(x^1,x^2,x^3\right)\neq(0,0,0)\right\},$$

where $\left(x^1,x^2,x^3\right)$ are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields

$$\mathbf{e}_1=e^{x^1}\frac{\partial}{\partial x^2},\quad \mathbf{e}_2=e^{x^1}\left(\frac{\partial}{\partial x^2}-\frac{\partial}{\partial x^3}\right),\quad \mathbf{e}_3=-\frac{\partial}{\partial x^1} \tag{2.4}$$

are linearly independent at each point of $\mathbb{P}$. Let g be the Riemannian metric defined by

$$\begin{aligned} g\left(\mathbf{e}_1,\mathbf{e}_1\right)&=g\left(\mathbf{e}_2,\mathbf{e}_2\right)=g\left(\mathbf{e}_3,\mathbf{e}_3\right)=1,\\ g\left(\mathbf{e}_1,\mathbf{e}_2\right)&=g\left(\mathbf{e}_2,\mathbf{e}_3\right)=g\left(\mathbf{e}_1,\mathbf{e}_3\right)=0.\end{aligned} \tag{2.5}$$

Let η be the 1-form defined by

$$\eta(Z)=g(Z,\mathbf{e}_3)\text{ for any }Z\in\chi(\mathbb{P}).$$

Let be the (1,1) tensor field defined by

$$\phi(\mathbf{e}_1)=\mathbf{e}_2,\ \phi(\mathbf{e}_2)=\mathbf{e}_1,\ \phi(\mathbf{e}_3)=0. \tag{2.6}$$

Then using the linearity of and g we have

$$\eta(\mathbf{e}_3)=1, \tag{2.7}$$

$$\phi^2(Z)=Z-\ \eta(Z)\mathbf{e}_3, \tag{2.8}$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W), \tag{2.9}$$

for any $Z, W \in \chi(\mathbb{P})$. Thus for $\mathbf{e}_3 = \xi$, (ϕ, ξ, η, g) defines an almost para-contact metric structure on $\mathbb{P}$.

Let ∇ be the Levi-Civita connection with respect to g. Then, we have

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1, \quad [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_2.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \end{aligned}$$

which is known as Koszul's formula.

Taking $\mathbf{e}_3 = \xi$ and using the Koszul's formula, we obtain

$$\begin{array}{lll} \nabla_{\mathbf{e}_1}\mathbf{e}_1 = -\mathbf{e}_3, & \nabla_{\mathbf{e}_1}\mathbf{e}_2 = 0, & \nabla_{\mathbf{e}_1}\mathbf{e}_3 = \mathbf{e}_1, \\ \nabla_{\mathbf{e}_2}\mathbf{e}_1 = 0, & \nabla_{\mathbf{e}_2}\mathbf{e}_2 = -\mathbf{e}_3, & \nabla_{\mathbf{e}_2}\mathbf{e}_3 = \mathbf{e}_2, \\ \nabla_{\mathbf{e}_3}\mathbf{e}_1 = 0, & \nabla_{\mathbf{e}_3}\mathbf{e}_2 = 0, & \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0. \end{array} \tag{2.10}$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_k, \quad R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l),$$

where the indices i, j, k and l take the values $1, 2$ and 3.

$$R_{122} = -\mathbf{e}_{1,}, \quad R_{133} = -\mathbf{e}_{1,}, \; R_{233} = -\mathbf{e}_2,$$

and

$$R_{1212} = R_{1313} = R_{2323} = 1. \tag{2.11}$$

3 Biharmonic Curves in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

Biharmonic equation for the curve γ reduces to

$$\nabla^3_{\mathbf{T}}\mathbf{T} - R(\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T})\mathbf{T} = 0, \tag{3.1}$$

that is, γ is called a biharmonic curve if it is a solution of the equation (3.1).

Let us consider biharmonicity of curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned}
&\nabla_{\mathbf{T}}\mathbf{T} = \kappa\mathbf{N},\\
&\nabla_{\mathbf{T}}\mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B},\\
&\nabla_{\mathbf{T}}\mathbf{B} = -\tau\mathbf{N},
\end{aligned} \tag{3.2}$$

where κ is the curvature of γ and τ its torsion and

$$\begin{aligned}
g(\mathbf{T},\mathbf{T}) &= 1,\ g(\mathbf{N},\mathbf{N}) = 1,\ g(\mathbf{B},\mathbf{B}) = 1,\\
g(\mathbf{T},\mathbf{N}) &= g(\mathbf{T},\mathbf{B}) = g(\mathbf{N},\mathbf{B}) = 0.
\end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$\begin{aligned}
&\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3,\\
&\mathbf{N} = N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3,\\
&\mathbf{B} = \mathbf{T}\times\mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3.254
\end{aligned} \tag{3.3}$$

Theorem 3.1. $\gamma : I \longrightarrow \mathbb{P}$ *is a biharmonic curve if and only if*

$$\begin{aligned}
&\kappa = \text{constant} \neq 0,\\
&\kappa^2 + \tau^2 = 1,\\
&\tau = \text{constant}.
\end{aligned} \tag{3.4}$$

Proof. Using (3.1) and Frenet formulas (3.2), we have (3.4).

Theorem 3.2. *All of biharmonic curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$ are helices.*

4 Evolute Curve of Biharmonic Curve in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

Definition 4.1. *Let unit speed curve $\gamma : I \longrightarrow \mathbb{P}$ and the curve $\beta : I \longrightarrow \mathbb{P}$ be given. For $\forall s \in I$, the tangent at the point $\beta(s)$ to the curve β passes through the tangent at the point $\gamma(s)$ and*

$$g(\mathbf{T}^*(s), \mathbf{T}(s)) = 0. \tag{4.1}$$

Then, β is called the evolute of the curve γ.

Let the Frenet-Serret frames of the curves γ and β be $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ and $\{\mathbf{T}^*,\mathbf{N}^*,\mathbf{B}^*\}$, respectively.

Theorem 4.2. *Let $\gamma : I \longrightarrow \mathbb{P}$ be a unit speed biharmonic curve and β its evolute curve on $\mathbb{P}$. Then, the parametric equations of β are*

$$\begin{aligned}
x_\beta^1(s) &= -s\cos\varphi + \tfrac{1}{\kappa^2}(-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2) \\
&\quad + \frac{1}{\kappa^2}\tan(\tau s+\zeta)\sin\varphi e^{-s\cos\varphi + C_1}\left(\sin[\Bbbk s + C] + \cos[\Bbbk s + C]\right)e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2} \\
&\quad .(-\Bbbk\sin\varphi\cos[\Bbbk s + C] + \cos\varphi\sin\varphi\sin[\Bbbk s + C]) \\
&\quad - \frac{1}{\kappa^2}\tan(\tau s+\zeta)\sin\varphi e^{-s\cos\varphi + C_1}\sin[\Bbbk s + C]\, e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}((\Bbbk\sin\varphi\sin[\Bbbk s + C] \\
&\quad + \cos\varphi\sin\varphi\cos[\Bbbk s + C]) + (-\Bbbk\sin\varphi\cos[\Bbbk s + C] + \cos\varphi\sin\varphi\sin[\Bbbk s + C])) + C_1,
\end{aligned}$$

$$\begin{aligned}
x_\beta^2(s) &= -\frac{\sin^3\varphi}{\kappa^2 - \sin^4\varphi}e^{-s\cos\varphi + C_1}([\Bbbk + \cos\varphi]\cos[\Bbbk s + C] + [-\Bbbk + \cos\varphi]\sin[\Bbbk s + C]) \\
&\quad + \frac{1}{\kappa^2}e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}(\Bbbk\sin\varphi\sin[\Bbbk s + C] + \cos\varphi\sin\varphi\cos[\Bbbk s + C]) \\
&\quad + \frac{1}{\kappa^2}e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}(-\Bbbk\sin\varphi\cos[\Bbbk s + C] + \cos\varphi\sin\varphi\sin[\Bbbk s + C]) \\
&\quad - \frac{1}{\kappa^2}\tan(\tau s+\zeta)(-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2)\sin\varphi e^{-s\cos\varphi + C_1}\sin[\Bbbk s + C] \\
&\quad + \frac{1}{\kappa^2}\tan(\tau s+\zeta)\cos\varphi e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}(-\Bbbk\sin\varphi\cos[\Bbbk s + C] + \cos\varphi\sin\varphi\sin[\Bbbk s + C]) + C_2,
\end{aligned}$$

$$\begin{aligned}
x_\beta^3(s) &= -\frac{\sin^3\varphi}{\kappa^2 - \sin^4\varphi}e^{-s\cos\varphi + C_1}(-\cos\varphi\cos[\Bbbk s + C] + [\Bbbk s + C]\sin[\Bbbk s + C]) \\
&\quad - \frac{1}{\kappa^2}e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}(-\Bbbk\sin\varphi\cos[\Bbbk s + C] + \cos\varphi\sin\varphi\sin[\Bbbk s + C]) \\
&\quad + \frac{1}{\kappa^2}\tan(\tau s+\zeta)\cos\varphi e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}((\Bbbk\sin\varphi\sin[\Bbbk s + C] + \cos\varphi\sin\varphi\cos[\Bbbk s + C]) \\
&\quad + (-\Bbbk\sin\varphi\cos[\Bbbk s + C] + \cos\varphi\sin\varphi\sin[\Bbbk s + C])) \\
&\quad + \frac{1}{\kappa^2}\tan(\tau s+\zeta)\sin\varphi e^{-s\cos\varphi + C_1}(\sin[\Bbbk s + C] + \cos[\Bbbk s + C])(-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2) + C_3,
\end{aligned} \tag{4.2}$$

where C, $\overline{C}_1$, $\overline{C}_2$, C_1, C_2, C_3 are constants of integration and $\Bbbk = \frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}$.

Proof. Since γ is biharmonic, γ is a helix. So, without loss of generality, we take the axis of γ is parallel to the vector $\mathbf{e}_3$. Then,

$$g(\mathbf{T}, \mathbf{e}_3) = T_3 = \cos\varphi, \tag{4.3}$$

where φ is constant angle.

The tangent of the curve β at the point $\beta(s)$ is the line constructed by the vector $\mathbf{T}^*(s)$.

The curve $\beta(s)$ may be given as

$$\beta(s) = \gamma(s) + \lambda\mathbf{N}(s) + \mu\mathbf{B}(s). \tag{4.4}$$

If we take the derivative (4.4), then we have

$$\beta'(s) = (1 - \lambda\kappa)\mathbf{T}(s) + \left(\lambda' - \mu\tau\right)\mathbf{N}(s) + \left(\lambda\tau + \mu'\right)\mathbf{B}(s). \tag{4.5}$$

Since the curve β is evolute of the curve γ, $g(\mathbf{T}^*(s), \mathbf{T}(s)) = 0$. Then, we get

$$\lambda = \frac{1}{\kappa}\ . \tag{4.6}$$

Using (4.5) and (4.6), we have

$$\beta'(s) = \left(\lambda' - \mu\tau\right)\mathbf{N}(s) + \left(\lambda\tau + \mu'\right)\mathbf{B}(s). \tag{4.7}$$

From the (4.4) and (4.7), the vector field β' is parallel to the vector field $\beta - \gamma$. Then, we have

$$\tau = \frac{\mu\lambda' - \mu'\lambda}{\mu^2 + \lambda^2} = \left[\arctan\left(-\frac{\mu}{\lambda}\right)\right]' = \text{constant}.$$

If we take the integral the last equation, we get

$$\arctan\left(-\frac{\mu}{\lambda}\right) = \tau s + \zeta, \tag{4.8}$$

where ζ is a constant of integration.

From (4.8), we obtain

$$\mu = -\frac{1}{\kappa}\tan(\tau s + \zeta). \tag{4.9}$$

The tangent vector can be written in the following form

$$\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3. \tag{4.10}$$

On the other hand the tangent vector $\mathbf{T}$ is a unit vector, so the following condition is satisfied

$$T_1^2 + T_2^2 = 1 - \cos^2\varphi. \tag{4.11}$$

Noting that $\cos^2\varphi + \sin^2\varphi = 1$, we have

$$T_1^2 + T_2^2 = \sin^2\varphi. \tag{4.12}$$

The general solution of (4.12) can be written in the following form

$$\begin{aligned} T_1 &= \sin\varphi\cos\mu, \\ T_2 &= \sin\varphi\sin\mu, \end{aligned} \tag{4.13}$$

where μ is an arbitrary function of s.

So, substituting the components T_1, T_2 and T_3 in the equation (4.7), we have the following equation

$$\mathbf{T} = \sin\varphi\cos\mu\mathbf{e}_1 + \sin\varphi\sin\mu\mathbf{e}_2 + \cos\varphi\mathbf{e}_3. \tag{4.14}$$

Since $|\nabla_{\mathbf{T}}\mathbf{T}| = \kappa$, we obtain

$$\mu = \frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}s + C, \tag{4.15}$$

where $C \in \mathbb{R}$.

Thus (4.14) and (4.15), imply

$$\mathbf{T} = \sin\varphi\cos\left[\Bbbk s + C\right]\mathbf{e}_1 + \sin\varphi\sin\left[\Bbbk s + C\right]\mathbf{e}_2 + \cos\varphi\mathbf{e}_3,$$

where $\Bbbk = \frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}$.

Using (2.4) in above equation, we obtain

$$\mathbf{T} = (-\cos\varphi, \sin\varphi e^{x^1}\left(\sin\left[\Bbbk s + C\right] + \cos\left[\Bbbk s + C\right]\right), \sin\varphi e^{x^1}\sin\left[\Bbbk s + C\right]). \tag{4.16}$$

From third component of $\mathbf{T}$, we have

$$\begin{aligned} \frac{dx^1}{ds} &= -\cos\varphi, \\ \frac{dx^2}{ds} &= \sin\varphi e^{-s\cos\varphi + C_1}\left(\sin\left[\Bbbk s + C\right] + \cos\left[\Bbbk s + C\right]\right), \\ \frac{dx^3}{ds} &= C_1\sin\varphi e^{-s\cos\varphi + C_1}\cos\left[\Bbbk s + C\right]. \end{aligned}$$

By direct calculations, we have

$$\begin{aligned} x^1(s) &= -s\cos\varphi + C_1, \\ x^2(s) &= C_2 - \frac{\sin^3\varphi}{\kappa^2-\sin^4\varphi} e^{-s\cos\varphi + C_1}([\mathbb{k}+\cos\varphi]\cos[\mathbb{k}s+C] \\ &\quad + [-\mathbb{k}+\cos\varphi]\sin[\mathbb{k}s+C]), \\ x^3(s) &= C_3 - \frac{\sin^3\varphi}{\kappa^2-\sin^4\varphi} e^{-s\cos\varphi + C_1}(-\cos\varphi\cos[\mathbb{k}s+C] \\ &\quad + [\mathbb{k}s+C]\sin[\mathbb{k}s+C]), \end{aligned} \tag{4.17}$$

where C_1, C_2, C_3 are constants of integration.

Using (4.10), we have

$$\nabla_{\mathbf{T}}\mathbf{T} = \left(T_1' + T_1T_3\right)\mathbf{e}_1 + \left(T_2' + T_2T_3\right)\mathbf{e}_2 + \left(T_3' - \left(T_1^2 - T_2^2\right)\right)\mathbf{e}_3. \tag{4.18}$$

From (3.1) and (5.11), we get

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \sin\varphi\left(-\mathbb{k}\sin[\mathbb{k}s+C] + \cos\varphi\cos[\mathbb{k}s+C]\right)\mathbf{e}_1 \\ &\quad + \sin\varphi\left(\mathbb{k}\cos[s+C] + \cos\varphi\sin[\mathbb{k}s+C]\right)\mathbf{e}_2 \\ &\quad - \sin^2\varphi\mathbf{e}_3, \end{aligned} \tag{4.19}$$

where $\mathbb{k} = \frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}$.

We substitute (4.9) and (4.6) into (4.4), we get

$$\beta(s) = \gamma(s) + \frac{1}{\kappa}\mathbf{N}(s) - \frac{1}{\kappa}\tan(\tau s + \zeta)\mathbf{B}(s). \tag{4.20}$$

By the use of Frenet formulas (4.2), we get

$$\begin{aligned} \mathbf{N} &= \frac{1}{\kappa}\nabla_{\mathbf{T}}\mathbf{T} \\ &= \frac{1}{\kappa}[(\mathbb{k}\sin\varphi\sin[\mathbb{k}s+C] + \cos\varphi\sin\varphi\cos[\mathbb{k}s+C])\mathbf{e}_1 \\ &\quad + (-\mathbb{k}\sin\varphi\cos[\mathbb{k}s+C] + \cos\varphi\sin\varphi\sin[\mathbb{k}s+C])\mathbf{e}_2 \\ &\quad - \sin^2\varphi\mathbf{e}_3]. \end{aligned} \tag{4.21}$$

Substituting (2.4) in (4.21), we have

$$\begin{aligned} \mathbf{N} = \frac{1}{\kappa}(&-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2. \\ &.e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(\mathbb{k}\sin\varphi\sin[\mathbb{k}s+C] + \cos\varphi\sin\varphi\cos[\mathbb{k}s+C]\right) \\ &+ e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(-\mathbb{k}\sin\varphi\cos[\mathbb{k}s+C] + \cos\varphi\sin\varphi\sin[\mathbb{k}s+C]\right) \\ &- e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(-\mathbb{k}\sin\varphi\cos[\mathbb{k}s+C] + \cos\varphi\sin\varphi\sin[\mathbb{k}s+C]\right)) \end{aligned} \tag{4.22}$$

where $\overline{C}_1, \overline{C}_2$ are constants of integration.

$$\mathbf{T} = (-\cos\varphi, \sin\varphi e^{x^1}\left(\sin\left[\Bbbk s + C\right] + \cos\left[\Bbbk s + C\right]\right), \sin\varphi e^{x^1}\sin\left[\Bbbk s + C\right]). \tag{4.23}$$

Noting that $\mathbf{T} \times \mathbf{N} = \mathbf{B}$, we have

$$\begin{aligned}\mathbf{B} =& \frac{1}{\kappa}(-\sin\varphi e^{-s\cos\varphi + C_1}\left(\sin\left[\Bbbk s + C\right] + \cos\left[\Bbbk s + C\right]\right) e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2} \\
& .\left(-\Bbbk\sin\varphi\cos\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\sin\left[\Bbbk s + C\right]\right) \\
& -\sin\varphi e^{-s\cos\varphi + C_1}\sin\left[\Bbbk s + C\right] e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}((\Bbbk\sin\varphi\sin\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\cos\left[\Bbbk s + C\right]) \\
& + \left(-\Bbbk\sin\varphi\cos\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\sin\left[\Bbbk s + C\right]\right)) \\
& .(-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2)\sin\varphi e^{-s\cos\varphi + C_1}\sin\left[\Bbbk s + C\right] \\
& -\cos\varphi e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(-\Bbbk\sin\varphi\cos\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\sin\left[\Bbbk s + C\right]\right) \\
& -\cos\varphi e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}((\Bbbk\sin\varphi\sin\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\cos\left[\Bbbk s + C\right]) \\
& + \left(-\Bbbk\sin\varphi\cos\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\sin\left[\Bbbk s + C\right]\right)) \\
& -\sin\varphi e^{-s\cos\varphi + C_1}\left(\sin\left[\Bbbk s + C\right] + \cos\left[\Bbbk s + C\right]\right)(-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2).\end{aligned} \tag{4.24}$$

Finally, we substitute (4.12), (4.17) and (4.24) into (4.20), we get (4.2). The proof is completed.

Corollary 4.3. *Let $\gamma : I \longrightarrow \mathbb{P}$ be a unit speed biharmonic curve and β its evolute curve on $\mathbb{P}$. Then, the parametric equations of γ are*

$$x^1(s) = -s\cos\varphi + C_1, \tag{4.25}$$

$$x^2(s) = C_2 - \frac{\sin^3\varphi}{\kappa^2 - \sin^4\varphi} e^{-s\cos\varphi + C_1}([\mathbb{k} + \cos\varphi]\cos[\mathbb{k}s + C] + [-\mathbb{k} + \cos\varphi]\sin[\mathbb{k}s + C]),$$

$$x^3(s) = C_3 - \frac{\sin^3\varphi}{\kappa^2 - \sin^4\varphi} e^{-s\cos\varphi + C_1}(-\cos\varphi\cos[\mathbb{k}s + C] + [\mathbb{k}s + C]\sin[\mathbb{k}s + C]),$$

where C_1, C_2, C_3 are constants of integration.

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