INSTITUTE OF PHYSICS PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **18** (2001) 5441–5454 PII: S0264-9381(01)27440-1

Scalar field spacetimes

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Received 1 August 2001, in final form 24 October 2001
Published 5 December 2001
Online at stacks.iop.org/CQG/18/5441

Abstract
Scalar field spacetimes are considered with a view towards their applications in cosmology. Some results existing in the literature are reviewed and some new results are proved concerning scalar field spacetimes in general and inhomogeneous G_2 cosmological models in particular.

PACS numbers: 0420J, 0440N, 9880

1. Introduction

Scalar field spacetimes have been widely used in general relativity (GR) to model certain situations of interest in both the astrophysical (see [1, 2] and the references cited therein) and the cosmological context [3], especially the so-called *inflationary scenarios* [4, 5].

In this paper we aim at studying some general features of scalar field spacetimes with a view towards their application as possible cosmological models at early stages in the evolution of the universe. In what follows, (M, g) will denote the spacetime, M being a connected four-dimensional Hausdorff manifold, and g a Lorentz metric which we take to be of signature + 2 and class C^2 at least (thus, M will be at least C^3). Greek indices will stand for coordinate indices and will run from 0 to 3, whereas lowercase Latin indices (also running from 0 to 3) will be tetrad indices associated with a pseudo-orthonormal tetrad $\{\theta^a\}$, i.e., $ds^2 = \eta_{ab}\theta^a\theta^b$. Einstein's field equations (EFEs) will be written in an arbitrary coordinate chart as $G_{\mu\nu} = T_{\mu\nu}$, where $G_{\mu\nu}$ and $T_{\mu\nu}$ denote the coordinate components of the Einstein and the energy–momentum tensors, respectively, the latter being assumed of the form

$$T_{\mu\nu} = \Phi_\mu\Phi_\nu - \left(\tfrac{1}{2}\Phi^\alpha\Phi_\alpha + V(\Phi)\right)g_{\mu\nu} \tag{1}$$

where $\Phi = \Phi(x^\alpha)$ is a real function on M, the *scalar field* $V(\Phi)$ is the *potential* describing some kind of self interaction, and $\Phi_\alpha = \Phi_{,\alpha}$ stand for the components of the gradient of Φ.

The paper is structured as follows. In section 2 we discuss some general features of scalar field spacetimes and prove some results concerning Segre classification, energy conditions and symmetries, especially isometries and homotheties, of this class of spacetimes. It is shown in particular that if a scalar field spacetime is to admit a homothety, then the potential must be of the exponential form or zero. Section 3 focuses on G_2 inhomogeneous models, giving the necessary and sufficient conditions for an orthogonally transitive G_2 spacetime to represent a scalar field such as the one presented above; this result is closely related to a similar one for perfect fluids given by Mars and Wolf [6]. In section 4 we approach the issue of inflation making some general remarks on the strong energy condition in various regions of the spacetime as well as the existence of privileged congruences of timelike curves, even in the case where the gradient of the scalar field is spacelike, thus suggesting a way of checking the inflationary character of a given scalar field spacetime, especially in the G_2-inhomogeneous case. In section 5 we give the field equations for separable orthogonally transitive G_2-models, discussing some issues concerning separability, classifying the various possible cases and writing the EFEs for each case as first-order autonomous systems of ordinary differential equations; all this is done in a completely general way, i.e. without any extra assumptions. In this respect, our work extends and generalizes some developments in this field already existing in the literature (see [7, 8] and the references cited therein). In section 2 we present two examples, one of them corresponding to self-similar solutions (i.e. solutions which admit a homothety). Finally, we summarize our results and relate them to others in the literature in section 7.

2. General considerations

In this section we aim at giving some general results regarding scalar field spacetimes, in particular those related to the algebraic classification of the energy–momentum tensor, energy conditions and symmetries (isometries and proper homotheties in particular).

Consider the energy–momentum tensor given by (1). It is immediately seen that Φ^α is always an eigenvector with corresponding eigenvalue $\lambda \equiv \frac{1}{2}\Phi^\alpha\Phi_\alpha - V(\Phi)$. Furthermore, λ is non-degenerate unless $\Phi^\alpha\Phi_\alpha = 0$, i.e. Φ^α is null. Assuming $\Phi^\alpha\Phi_\alpha \neq 0$, it follows that three other eigenvectors exist with (degenerate) eigenvalue $\sigma \equiv -\frac{1}{2}\Phi^\alpha\Phi_\alpha - V(\Phi)$, and therefore the Segre type [9] of the energy–momentum tensor above is $\{1, (111)\}$ in the spacetime region where Φ^α is timelike, or $\{(1, 11)1\}$ wherever Φ^α is spacelike. If Φ^α is null (and, by continuity, this must necessarily be so at some region in M if the spacetime is to contain regions of the two previous types discussed above), then the Segre type is $\{(2, 11)\}$ with degenerate eigenvalue $\nu \equiv -V(\Phi)$. The Segre type $\{(1, 111)\}$ can also occur at points in M, and then one has $\Phi^\alpha\Phi_\alpha = 0$ and $V(\Phi) =$ constant as follows from the contracted Bianchi identity, and the energy–momentum tensor is then a Λ-term [9], this case corresponds to $\Phi =$ constant.

Following Senovilla and Vera [10, 11], we see that (the open submanifold of) a spacetime M, whose matter content is described by an energy–momentum tensor of the form (1), can be decomposed (in an obvious notation) as the disjoint union of the following sets:

$$M = \{1, (111)\} \bigcup \{(1, 11)1\} \bigcup \mathcal{F} \tag{2}$$

where $\{1, (111)\}$ and $\{(1, 11)1\}$ are open sets, and $\mathcal{F} \equiv \{(2, 11)\} \bigcup \{(1, 111)\}$ is a closed set which has no interior whenever the energy–momentum tensor is analytic (it can be interpreted as some kind of border between $\{1, (111)\}$ and $\{(1, 11)1\}$).

Conditions implied by the dominant energy condition (DEC) can be almost immediately read off from the expression of $T_{\mu\nu}$ in (1) in terms of an adapted orthonormal tetrad (cases $\{1, (111)\}$ and $\{(1, 11)1\}$) one of whose members is proportional to Φ^α or a real null tetrad with one of the null vectors also parallel to Φ^α; following [12] (see also [9]) we then have

Segre type	$\{1,\ (111)\}$	$\{(1,\ 11)1\}$	$\{(2,\ 11)\}$	$\{(1,\ 111)\}$
DEC	$-\lambda \geqslant 0,\ V(\Phi) \geqslant 0$	$-\sigma \geqslant 0,\ V(\Phi) \geqslant 0$	$V(\Phi) \geqslant 0$	$V(\Phi) = \text{constant} \geqslant 0$

From the values of λ and σ, and the signs of $\Phi^\alpha\Phi_\alpha$ in the relevant regions (i.e. $\Phi^\alpha\Phi_\alpha < 0$ in $\{1, (111)\}$ and $\Phi^\alpha\Phi_\alpha > 0$ in $\{(1, 11)1\}$) it follows that the DEC is satisfied all over M, if and only if $V(\Phi) \geqslant 0$.

In the next proposition, we summarize some results which hold if the spacetime admits an isometry or a homothety, generated respectively by a Killing vector (KV) or a homothetic vector (HV), $\vec{\xi}$, i.e.,

$$\mathcal{L}_{\vec{\xi}}\, g_{\mu\nu} = 2\,k\,g_{\mu\nu} \tag{3}$$

where k is a constant that vanishes in the case of a KV, and is non-zero if $\vec{\xi}$ is a (proper) HV (in which case one can always re-scale $\vec{\xi}$ so as to set $k = 1$). Note that in both cases (KV or proper HV)

$$\mathcal{L}_{\vec{\xi}}\, T_{\mu\nu} = 0. \tag{4}$$

Theorem 1. *Let (M, g) be a scalar field spacetime admitting an HV, $\vec{\xi}$; it then follows:*

(1) $\mathcal{L}_{\vec{\xi}}\Phi_\alpha = 0$ $\left(\text{hence } \mathcal{L}_{\vec{\xi}}\Phi^\alpha = -2k\Phi^\alpha\right)$.
(2) $\mathcal{L}_{\vec{\xi}} V(\Phi) = -2kV(\Phi)$.
(3) If $\vec{\xi}$ is a proper HV, the only possible form for the potential $V(\Phi)$ is $V(\Phi) = \Lambda\exp(\kappa\Phi)$, where Λ and $\kappa (\neq 0)$ are real constants, and $\mathcal{L}_{\vec{\xi}}\Phi = -2k/\kappa$.

Proof. Consider first the case $\Phi^\alpha\Phi_\alpha \neq 0$, it follows that λ is a non-degenerate eigenvalue corresponding to the eigenvector Φ^α; one has

$$\mathcal{L}_{\vec{\xi}}\Phi^\alpha = a\Phi^\alpha + Z^\alpha \qquad \text{with} \qquad \Phi^\alpha Z_\alpha = 0 \qquad \text{hence} \qquad \mathcal{L}_{\vec{\xi}}\Phi_\alpha = (2k + a)\Phi_\alpha + Z_\alpha \tag{5}$$

where a is in principle an arbitrary function; thus

$$\mathcal{L}_{\vec{\xi}}\left(T_{\mu\alpha}\Phi^\alpha\right) = T_{\mu\alpha}(a\Phi^\alpha + Z^\alpha) = a\lambda\Phi_\mu + T_{\mu\alpha}Z^\alpha \tag{6}$$

and also

$$\mathcal{L}_{\vec{\xi}}\left(T_{\mu\alpha}\Phi^\alpha\right) = \mathcal{L}_{\vec{\xi}}(\lambda\Phi_\mu) = \mathcal{L}_{\vec{\xi}}(\lambda)\Phi_\mu + \lambda(2k + a)\Phi_\mu + \lambda Z_\mu. \tag{7}$$

Equating the last two equations, simplifying and contracting with Φ^μ it follows that

$$(\mathcal{L}_{\vec{\xi}}\lambda + 2k\lambda)\Phi_\mu\Phi^\mu = 0 \qquad \text{i.e.} \qquad \mathcal{L}_{\vec{\xi}}\lambda = -2k\lambda. \tag{8}$$

Substituting (8) into (7) and equating with (6) it follows that $T_{\mu\alpha}Z^\alpha = \lambda Z_\mu$. Now, since λ is non-degenerate[1], it must be that $Z^\alpha = 0$ and therefore

$$\mathcal{L}_{\vec{\xi}}\Phi^\alpha = a\Phi^\alpha. \tag{9}$$

Now one has $\mathcal{L}_{\vec{\xi}}\Phi_\alpha = (2k + a)\Phi_\alpha$ and taking into consideration the expression (1) for the energy–momentum tensor, computing explicitly $\mathcal{L}_{\vec{\xi}}T_{\mu\alpha}$ and equating it to zero one easily gets

$$a = -2k \qquad \text{i.e.} \qquad \mathcal{L}_{\vec{\xi}}\Phi_\alpha = 0 \qquad \text{and also} \qquad \mathcal{L}_{\vec{\xi}}V = -2kV. \tag{10}$$

[1] Note that all of the previous results still hold for an arbitrary tensor with an eigenvector corresponding to an eigenvalue λ, if λ were a degenerate eigenvalue, (8) would still hold. But then the Lie derivative of the associated eigenvector would be an eigenvector of eigenvalue λ, and not necessarily the same one; i.e., (9) would not hold in general.

Now, $(\mathcal{L}_{\vec{\xi}}\Phi)_{,\beta} = 0$ as follows directly from (10), that is, $\mathcal{L}_{\vec{\xi}}\Phi = \alpha \in \mathbb{R}$; substituting (10) it follows that

$$\mathcal{L}_{\vec{\xi}}V = \frac{\mathrm{d}V}{\mathrm{d}\Phi}\mathcal{L}_{\vec{\xi}}\Phi = \frac{\mathrm{d}V}{\mathrm{d}\Phi}\alpha = -2kV \qquad \text{i.e.} \quad V(\Phi) = \Lambda \exp\left(-2\frac{k}{\alpha}\Phi\right). \tag{11}$$

Note that $\alpha = 0$ (whenever $k \neq 0$, i.e. the HV is proper) trivially implies $V = 0$. The case when $\Phi^\alpha\Phi_\alpha = 0$ follows much along the same lines. □

Thus, if a scalar field spacetime is self-similar (that is, if it admits a proper homothety), the potential *must* necessarily be of the exponential type (including zero as a special case). Thus, in particular, one cannot have potentials of the form/containing terms m^2. This result can also be easily generalized to the case of complex scalar fields.

3. Inhomogeneous models

In this section we will consider spacetimes admitting an Abelian group G_2 of isometries acting on two-dimensional, spacelike orbits S_2. Such spacetimes were classified, and normal forms for the allowed line elements obtained by Wainwright [13]. Subsequently, this family, and in particular the subclass of orthogonally transitive Abelian metrics[2] (OT-G_2 for short), has been investigated by many authors as suitable cosmological models describing inhomogeneous universes. Most studies have been done assuming perfect fluid content, and a formalism based on the theory of dynamical systems has been set up in order to treat the family as a whole, insisting in the qualitative behaviour of the solutions rather than on their character as exact solutions of the EFEs. In so doing, the importance of self-similar models (i.e. those admitting an HV) has become increasingly clear, as they may describe the asymptotic behaviour of the universe at both early and late times of some other, more general, models; the reader is referred to [14, 15] and references cited therein for the details. More recently, some metrics belonging to this class of inhomogeneous cosmologies have been investigated assuming a matter content of the scalar field type, as the one assumed here (see [3, 7, 8] and also [16, 17]).

In this section, we intend to discuss certain general features of orthogonally transitive G_2-cosmologies assuming a matter content of the scalar field type (energy–momentum tensor given by (1)), and relate them to various results existing in the literature. As is customary, we shall designate the two commuting KV spanning the G_2 as $\vec{\xi}$ and $\vec{\eta}$, and use coordinates adapted to them, say y and z, respectively (that is: $\vec{\xi} = \partial_y$ and $\vec{\eta} = \partial_z$), the metric thus becoming independent of these coordinates; we shall choose two further coordinates, t and x, to complete the chart.

Mars [18] (see also Mars and Wolf [6]) proved some general results concerning OT-G_2 perfect fluids; thus, working with a pseudo-orthonormal tetrad adapted to the symmetries of the problem (i.e. θ_0, θ_1 in the t, x plane: surfaces orthogonal to the group orbits and θ_2, θ_3 on the orbits S_2), they showed that the EFEs for a perfect fluid are equivalent to the following set of equations (note that all indices are tetrad indices in the following equations):

$$(G_{00} + G_{22})(G_{11} - G_{22}) - G_{01}^2 = 0 \tag{12}$$

$$G_{22} - G_{33} = 0 \qquad\qquad G_{23} = 0 \tag{13}$$

plus the following two inequalities:

$$G_{00} - G_{11} + 2G_{22} \neq 0 \qquad\qquad \frac{G_{00} + G_{22}}{G_{00} - G_{11} + 2G_{22}} > 0. \tag{14}$$

[2] That is, whenever the orbits S_2 admit orthogonal surfaces.

Further, they showed that if a given OT-G_2 spacetime satisfied equations (12) and (13), then the metric obtained from that reversing the roles of the coordinates t and x, would also satisfy (12) and (13). Within this formalism, it is easy to show that:

Theorem 2. *The necessary and sufficient condition for an OT-G_2 spacetime to represent a scalar field with energy momentum-tensor given by (1) is that equations (12) and (13), and*

$$\left[\theta_t^0\,(G_{00}+G_{22})^{1/2}+\theta_t^1\,(G_{11}-G_{22})^{1/2}\right]_{,x}=\left[\theta_x^0\,(G_{00}+G_{22})^{1/2}+\theta_x^1\,(G_{11}-G_{22})^{1/2}\right]_{,t} \quad (15)$$

are satisfied.

Proof. We begin by noting that any vector field orthogonal to the group orbits S_2 and invariant under G_2 is hypersurface orthogonal. Since these two conditions together imply that the associated one-form, say u must be of the form: $u_\mu \mathrm{d}x^\mu = A(t,x)\mathrm{d}t + B(t,x)\mathrm{d}x$ whence $u_{[\mu,\nu}u_{\lambda]} = 0$, it then follows that $u_\mu \propto \Phi_{,\mu}$ with $\Phi = \Phi(t,x)$. Next, one recalls that any unit, non-degenerate eigenvector of $T_{\mu\nu}$ is invariant under G_2, this follows from equations (9) and (10) in theorem 1 if one demands that $\vec{\xi}$ is a KV, that is, $k = 0$. Computing then the tetrad components of the Einstein tensor for an OT-G_2, one readily gets that $G_{02} = G_{03} = G_{12} = G_{13} = 0$, i.e., if represented in matrix form it has box-diagonal structure, now (13) implies a double degeneracy in the eigenvalues corresponding to eigenvectors tangent to the group orbits S_2 (the repeated eigenvalue being G_{22}), whereas (12) implies that G_{22} is also a real eigenvalue corresponding to a (real) eigenvector orthogonal to S_2, that is, the Segre type is one of the following: $\{1,(111)\}$, $\{(1,11)1\}$ or $\{(2,11)\}$ (or the only possible degeneracy of these types, that is $\{(1,111)\}$), and consequently there exists a unit or null eigenvector tangent to the surfaces orthogonal to the group orbits, which on account of the foregoing discussion is proportional to a gradient and invariant under G_2 (see footnote 3), that is

$$T_{\mu\nu} = \sigma\Phi_{,\mu}\Phi_{,\nu} + G_{22}g_{\mu\nu} \quad (16)$$

where σ is some function of the coordinates. For $T_{\mu\nu}$ to be of the required form, we need $\sigma = 1$ to begin with[4]. Putting now $\sigma = 1$ above and working in tetrad components we get from the field equations

$$G_{00} = \Phi_0^2 - G_{22} \qquad G_{01} = \Phi_0\Phi_1 \qquad \text{and} \qquad G_{11} = \Phi_1^2 + G_{22} \quad (17)$$

which imply $\Phi_0 = \sqrt{G_{00}+G_{22}}$ and $\Phi_1 = \sqrt{G_{11}-G_{22}}$, taking now into account that $\Phi_t = \theta_t^0\Phi_0 + \theta_t^1\Phi_1$ and $\Phi_x = \theta_x^0\Phi_0 + \theta_x^1\Phi_1$, the integrability conditions for $\Phi(t,x)$ then take the form of (15).

Putting next, without loss of generality $G_{22} = -\left(\frac{1}{2}\Phi^{,\alpha}\Phi_{,\alpha} + V\right)$ where V is in principle a function of the coordinates, the contracted Bianchi identity implies

$$(\Box\,\Phi)\Phi_{,\mu} - V_{,\mu} = 0 \quad (18)$$

hence $V = V(\Phi)$ as required (or else $\Box\,\Phi = V_{,\alpha} = 0$, namely, a massless scalar field plus a Λ-term). □

The above result shows that in order to consider scalar field OT-G_2 exact solutions, it suffices to deal with equations (12), (13) and (15), the full set of field equations then providing expressions for the scalar field Φ and the potential $V(\Phi)$.

[3] This holds in the degenerate cases as well, as it follows from the eigenvector equation specialized to eigenvectors, orthogonal to the group orbits and taking into account that the corresponding eigenvalues are invariant under G2.

[4] In fact, what is required is $\sigma = \sigma(\Phi)$, or equivalently $\sigma_{,\alpha} \propto \Phi_{,\alpha}$; but if that is the case, we can always, by means of a suitable redefinition of Φ put $\sigma = 1$. In any case, note that $\sigma = \sigma(t,x)$ as a consequence of the invariance under the isometry group G_2 of $T_{\mu\nu}$, $g_{\mu\nu}$ and the functions G_{22} and Φ.

Note that, if the line element and pseudo-orthonormal tetrad are chosen as in (25) and (27), then (15) reads simply as

$$\left[e^{f_1}\,(G_{00}+G_{22})^{1/2}\right]_{,x} = \left[e^{f_1}\,(G_{11}-G_{22})^{1/2}\right]_{,t}. \tag{19}$$

Further, the character of $\Phi_{,\alpha}$ (timelike, spacelike or null) can be easily read from the sign of the expression $G_{11} - G_{00} - 2G_{22} = \Phi_1^2 - \Phi_0^2 = \Phi_\gamma \Phi^\gamma$, thus if $G_{11} - G_{00} - 2G_{22} < 0$ over a certain spacetime region, it follows that $\Phi_{,\alpha}$ is timelike there and consequently that region is in $\{1,(111)\}$. Similarly, $G_{11} - G_{00} - 2G_{22} > 0$ implies that $\Phi_{,\alpha}$ is spacelike and the region where this holds is in $\{(1,11)1\}$ whereas $G_{11} - G_{00} - 2G_{22} = 0$ implies that the region under consideration is $\mathcal{F} = \{(2,11)\} \bigcup \{(1,111)\}$; more precisely, $\{(2,11)\}$ if $G_{01} \neq 0$ and $\{(1,111)\}$ whenever $G_{01} = 0$.

4. Inflation

A given cosmological model is said to inflate if there exists some privileged timelike congruence, of tangent vector $\vec{u}$, whose deceleration parameter q is negative, q being defined as

$$q = -3\Theta^2\left(\dot{\Theta} + \tfrac{1}{3}\Theta^2\right) \tag{20}$$

where $\Theta \equiv u^\alpha_{;\alpha}$ is the expansion of the congruence and $\dot{\Theta} \equiv \Theta_{,\alpha}u^\alpha$. This can always be made sense of in the case of spatially homogeneous models, since there is then a naturally well-defined timelike congruence, namely the integral lines of the field of normals to the group orbits or surfaces of homogeneity.

In the case of (spatially) inhomogeneous models there does not seem to be an '*a priori*' natural way of seeing whether the models inflate or not, as the hypersurfaces Φ = constant are not necessarily spacelike. Nevertheless, in most of the explicit inhomogeneous scalar field cosmologies considered in the literature (see for instance [16, 17]), it so happened that the gradient of the scalar field was timelike everywhere and one could then consider $u^\alpha = \Phi^\alpha\left(-\Phi_\gamma\Phi^\gamma\right)^{-1/2}$ as the unit timelike vector field tangent to the 'privileged' timelike congruence, performing all the calculations with respect to u^α and then it was found that some of the models considered would inflate while some others did not (in fact, these models are completely equivalent to perfect fluid ones with pressure and density given by $p = -\left(\frac{1}{2}\Phi_\gamma\Phi^\gamma + V(\Phi)\right)$ and $\mu = -\frac{1}{2}\Phi_\gamma\Phi^\gamma + V(\Phi)$, respectively). In some other cases, though, this interpretation is not possible in general (see [7]), and it has then been suggested that one should look at the fulfilment of the strong energy condition, as its violation is a necessary condition for inflation to occur [19].

In this section we will try to offer a complementary point of view regarding the issue of inflation in inhomogeneous scalar field spacetimes, based in the Segre decomposition of the spacetime manifold that, following [11], we put forward in section 2.

To this end, recall the Segre decomposition of the manifold given by (2), we had: $M = \{1,(111)\} \bigcup \{(1,11)1\} \bigcup \mathcal{F}$ with $\mathcal{F} \equiv \{(2,11)\} \bigcup \{(1,111)\}$ being a closed set (with no interior in the case of analytic energy–momentum tensor). We now concentrate on the two open components of M, namely $\{1,(111)\}$ and $\{(1,11)1\}$; which we shall designate from now on as the T-region and S-region, since the gradient of the scalar field is timelike and spacelike, respectively. In both of these regions the energy–momentum tensor T is diagonal and we can then choose a tetrad, say $\{\theta_0,\theta_1,\theta_2,\theta_3\}$, θ_0 being its timelike member so that T can be written as:

$$T = \mu\theta_0\otimes\theta_0 + p_1\theta_1\otimes\theta_1 + p_2\theta_2\otimes\theta_2 + p_3\theta_3\otimes\theta_3 \tag{21}$$

where $\mu = -\frac{1}{2}\Phi_\gamma\Phi^\gamma + V(\Phi)$ and $p_1 = p_2 = p_3 = -\left(\frac{1}{2}\Phi_\gamma\Phi^\gamma + V(\Phi)\right)$ in the T-region (with $\theta_{0\alpha} = \Phi_\alpha\left(-\Phi_\gamma\Phi^\gamma\right)^{-1/2}$); and $\mu = -p_1 = -p_2 = \frac{1}{2}\Phi_\gamma\Phi^\gamma + V(\Phi)$, $p_3 = \frac{1}{2}\Phi_\gamma\Phi^\gamma - V(\Phi)$ in the S-region; now the strong energy condition implies that the trace of the energy–momentum tensor should be positive[5], i.e., $\mu + \sum_i p_i \geqslant 0$, and it is immediately seen that in the T-region one has

$$\mu + \sum_i p_i = -2\Phi_\gamma\Phi^\gamma - 2V(\Phi) \geqslant 0 \Leftrightarrow V(\Phi) \leqslant -\Phi_\gamma\Phi^\gamma \tag{22}$$

whereas in the S-region one has

$$\mu + \sum_i p_i = -2V(\Phi) \tag{23}$$

that is, the strong energy condition is generically violated in this region (unless $V(\Phi) = 0$).

Next we note that in the case of G_2 models, there is an 'almost everywhere' privileged timelike congruence on M (that is: in the two regions T and S, since then and provided that the Ricci tensor is analytic, $\mathcal{F}$ has an empty interior); namely: in the T-region the one associated to the timelike eigenvector of the energy–momentum tensor: θ_0^α, which in this case is unique and coincides with $u^\alpha = \Phi^\alpha\left(-\Phi_\gamma\Phi^\gamma\right)^{-1/2}$ (thus allowing for the perfect fluid interpretation alluded to above); in the S-region θ_0^α is no longer unique as there is a whole hypersurface of timelike eigenvectors; however, there is a single timelike eigendirection amongst them such that it is orthogonal to the group orbits and also to the gradient of the scalar field $\Phi_{,\alpha}$. This seems to suggest that the inflationary character of this type of solutions should be checked relatively to those timelike congruences, each one in the relevant region.

If one is interested in the kind of models we will be considering in the next section (namely: separable *B-ii* metrics in Wainwright's classification), it is very easy to show that in the S-region, the preferred timelike vector θ_0^α must be:

$$\vec{\theta}_0 = \mathrm{e}^{-T_1 - X_1}\left[-\frac{h(x)}{\sqrt{h(x)^2 - f(t)^2}}\partial_t + \frac{f(t)}{\sqrt{h(x)^2 - f(t)^2}}\partial_x\right] \tag{24}$$

where $f(t) = \Phi(t,x)_{,t}$ and $h(x) = \Phi(t,x)_{,x}$; as it follows from the block-diagonal structure of the Einstein tensor, the timelike character of $\vec{\theta}_0$ and the fact that it is orthogonal to $\Phi_{,\alpha}$ (see the next section). In the general case, though, one cannot decide whether q is positive or negative, so that this has to be checked for particular solutions.

5. Diagonal, separable OT-G_2 cosmologies

In this section we shall look into a class of OT-G_2 scalar field cosmologies, namely those which are diagonal (type *B-ii* in Wainwright's classification [13]) and separable in coordinates adapted to the Killing vectors. Separability of the metric functions in a given context is a concept which can be given a precise and invariant definition (see for instance [18]); in the current context this simply means that the line element can be written as

$$\mathrm{d}s^2 = \exp(2f_1)(-\mathrm{d}t^2 + \mathrm{d}x^2) + \exp(f_2)(\exp(2f_3)\mathrm{d}y^2 + \exp(-2f_3)\mathrm{d}z^2) \tag{25}$$

where

$$f_A(t,x) = T_A(t) + X_A(x) \qquad A = 1, 2, 3. \tag{26}$$

This family of solutions contains a distinguished subclass formed by the self-similar metrics (i.e. those admitting an HV), whose interest as asymptotic cosmological models,

[5] In fact, the S.E.C. states that $\mu + \sum_i p_i \geqslant 0$ and also $\mu + p_i \geqslant 0$, but the second inequality is supposed to hold as a consequence of the dominant energy condition, which has been assumed to hold throughout (see section 2).

whenever representing a perfect fluid, has already been mentioned in a previous section. See [15] for G_2 solutions admitting an HV.

Perfect fluid solutions in Wainwright's class *B-ii* which are separable in co-moving coordinates are known to a large extent (see [20–22], and also [10]), and a dynamical systems analysis of this class can be found in [14] using dimensionless, expansion normalized, variables. As for perfect fluid solutions of the above class (i.e. separable in coordinates adapted to the KV, metric as in (25)), they have been extensively dealt with in [10].

In the case of scalar field content, Olasagasti [3], and Ibáñez and Olasagasti [7, 8], have looked from the dynamical systems point of view into two separate families, both of them of the above form (25), showing that the phase space can be written, in both cases, in terms of new variables in such a way that it becomes compact, performing next the usual qualitative analysis of the solutions (fixed points, invariant subspaces, etc). In order to write the EFE as a dynamical system, the authors define new dimensionless variables, which consist in first-order time derivatives of metric functions, normalized by the 'expansion'[6] of the timelike congruence associated with the time coordinate t, and make some 'ad hoc' simplifying assumptions, such as separability of the scalar field, exponential form of the potential, etc.

It is our aim to deal with the problem in full generality; that is without making any of the above simplifying assumptions or restricting ourselves to a particular family within this class of separable *B-ii* metrics. In so doing, we will show that some of the assumptions in [3], and [7, 8] were not such, but rather followed from more general, previous assumptions, and also we will extend and complete their analysis.

5.1. Field equations: separability of the scalar field

Following [23] we shall choose the following tetrad adapted to the geometry of the problem:

$$\theta^0 = e^{f_1}\,dt \quad \theta^1 = e^{f_1}\,dx \quad \theta^2 = e^{f_2/2}e^{f_3}\,dy \quad \theta^3 = e^{f_2/2}e^{-f_3}\,dz \tag{27}$$

the Einstein tensor components are in this tetrad:

$$G_{00} = e^{-2f_1}\left[\tau_1\tau_2 + \tfrac{1}{4}\tau_2^2 - \tau_3^2 - \lambda_2' - \tfrac{3}{4}\lambda_2^2 - \lambda_3^2 + \lambda_1\lambda_2\right] \tag{28}$$

$$G_{01} = e^{-2f_1}\left[\tau_1\lambda_2 + \tau_2\left(\lambda_1 - \tfrac{1}{2}\lambda_2\right) - 2\tau_3\lambda_3\right] \tag{29}$$

$$G_{11} = e^{-2f_1}\left[\tau_1\tau_2 - \tfrac{3}{4}\tau_2^2 - \tau_3^2 - \dot\tau_2 + \lambda_1\lambda_2 + \tfrac{1}{4}\lambda_2^2 - \lambda_3^2\right] \tag{30}$$

$$G_{22} = e^{-2f_1}\Big[\lambda_1' + \tfrac{1}{2}\lambda_2' - \lambda_3' + \tfrac{1}{4}\lambda_2^2 - \lambda_2\lambda_3 + \lambda_3^2 - \dot\tau_1 - \tfrac{1}{2}\dot\tau_2 + \dot\tau_3 - \tfrac{1}{4}\tau_2^2 + \tau_2\tau_3 - \tau_3^2\Big] \tag{31}$$

$$G_{33} = e^{-2f_1}\Big[\lambda_1' + \tfrac{1}{2}\lambda_2' + \lambda_3' + \tfrac{1}{4}\lambda_2^2 + \lambda_2\lambda_3 + \lambda_3^2 - \dot\tau_1 - \tfrac{1}{2}\dot\tau_2 - \dot\tau_3 - \tfrac{1}{4}\tau_2^2 - \tau_2\tau_3 - \tau_3^2\Big] \tag{32}$$

where $\tau_A \equiv \dot T_A,\ \lambda_A \equiv X_A'$, for $A = 1, 2, 3$ and dots and primes stand for derivatives with respect to t and x, respectively. Now, $G_{22} = G_{33}$ readily implies

$$\dot\tau_3 + \tau_2\tau_3 = \lambda_3' + \lambda_2\lambda_3 = K \tag{33}$$

where K is some real constant.

On the other hand, the tetrad components of $\Phi_{,\mu}$ are $\Phi_0 = e^{-f_1}\Phi_{,t}$, $\Phi_1 = e^{-f_1}\Phi_{,x}$ and $\Phi_2 = \Phi_3 = 0$; and the field equations then imply:

$$\Phi_0^2 = G_{00} + G_{22} \qquad \Phi_1^2 = G_{11} - G_{22} \qquad V(\Phi) = \tfrac{1}{2}(G_{00} + G_{11}) \tag{34}$$

[6] We use quotation marks because the term *expansion* usually refers to a unit (or null) vector field, whereas in this case it stands for the divergence of the vector field ∂_t. Note that in the above references, the authors invoke 'practical reasons' for the use of such normalization.

from where it follows:

$$\Phi_{,t}^2 = -\dot{\tau}_1 - \tfrac{1}{2}\dot{\tau}_2 - 2\tau_3^2 + \tau_1\tau_2 + \lambda_1' - \tfrac{1}{2}\lambda_2' - \tfrac{1}{2}\lambda_2^2 + \lambda_1\lambda_2 \tag{35}$$

$$\Phi_{,x}^2 = \dot{\tau}_1 - \tfrac{1}{2}\dot{\tau}_2 - \tfrac{1}{2}\tau_2^2 + \tau_1\tau_2 - \lambda_1' - \tfrac{1}{2}\lambda_2' - 2\lambda_3^2 + \lambda_1\lambda_2 \tag{36}$$

thus,

$$\Phi_{,t} = \sqrt{a(t) + b(x)} \qquad \text{and} \qquad \Phi_{,x} = \sqrt{p(t) + q(x)} \tag{37}$$

the integrability conditions for Φ then demanding:

$$\frac{b'(x)}{\sqrt{a(t) + b(x)}} = \frac{\dot{p}(t)}{\sqrt{p(t) + q(x)}} \tag{38}$$

and there are then the following three possibilities:

(1) $b(x) = \text{constant}$ and $p(t) = \text{constant}$, and therefore $\Phi_{,t} = f(t)$ and $\Phi_{,x} = h(x)$; that is $\Phi(t, x) = \Phi_1(t) + \Phi_2(x)$ and the scalar field is then separable in the two variables t and x.

(2) $\Phi_{,t} = t^{-1}\sqrt{k + (2c)^{-1}t^2x^2}$ and $\Phi_{,x} = x^{-1}\sqrt{k + (2c)^{-1}t^2x^2}$, with k, c constants. The scalar field is then $\Phi(t, x) = \sqrt{k + (2c)^{-1}t^2x^2} - \sqrt{2k}\arctan\left(\sqrt{k}/\sqrt{k + (2c)^{-1}t^2x^2}\right)$; i.e. non-separable.

(3) $\Phi_{,t} = m^{-1}\sqrt{mn^2t + m^2nx + r}$ and $\Phi_{,x} = n^{-1}\sqrt{mn^2t + m^2nx + r}$, where m, n and r are constants. The scalar field is then $\Phi(t, x) = (2/3)m^{-2}n^{-2}(mn^2t + m^2nx + r)^{3/2}$, non-separable too.

A rather tedious and lengthy calculation shows now that the last two cases are not compatible with the field equations (34), therefore concluding that the scalar field must be separable. Note that in previous works (see [7, 8, 16, 17]) and the references cited therein) separability of the scalar field was actually *assumed* as an extra (reasonable) requirement; this shows that it follows in fact from the assumed separability of the metric coefficients in coordinates adapted to the Killing vectors. Thus, from now on we shall write

$$\Phi_{,t} = f(t) \qquad \text{and} \qquad \Phi_{,x} = h(x) \tag{39}$$

also note that from equations (35) and (36) above we have

$$\left(\lambda_1 - \tfrac{1}{2}\lambda_2\right)' + \lambda_2\left(\lambda_1 - \tfrac{1}{2}\lambda_2\right) = -k_1 \tag{40}$$

$$\left(\tau_1 - \tfrac{1}{2}\tau_2\right)^{\cdot} + \tau_2\left(\tau_1 - \tfrac{1}{2}\tau_2\right) = k_2 \tag{41}$$

with k_1, k_2 constants. Taking all this into account, we can rewrite the field equations in the following form:

$$\dot{\tau}_1 = \tfrac{1}{4}\tau_2^2 - \tau_3^2 - \tfrac{1}{2}f^2 - \tfrac{1}{2}(k_1 - k_2) \tag{42}$$

$$\dot{\tau}_2 = 2\tau_1\tau_2 - 2\tau_3^2 - \tfrac{1}{2}\tau_2^2 - f^2 - (k_1 + k_2) \tag{43}$$

$$\dot{\tau}_3 + \tau_2\tau_3 = K \tag{44}$$

$$\lambda_1' = \tfrac{1}{4}\lambda_2^2 - \lambda_3^2 - \tfrac{1}{2}h^2 - \tfrac{1}{2}(k_1 - k_2) \tag{45}$$

$$\lambda_2' = 2\lambda_1\lambda_2 - 2\lambda_3^2 - \tfrac{1}{2}\lambda_2^2 - h^2 + (k_1 + k_2) \tag{46}$$

$$\lambda_3' + \lambda_2\lambda_3 = K \tag{47}$$

$$\dot{\tau}_2 + \tau_2^2 - \left(\lambda_2' + \lambda_2^2\right) = 2V(\Phi)\mathrm{e}^{2f_1} \tag{48}$$

$$\tau_1\lambda_2 + \tau_2\left(\lambda_1 - \tfrac{1}{2}\lambda_2\right) - 2\tau_3\lambda_3 = fh. \tag{49}$$

Thus, the system becomes decoupled in the t and x coordinates, and the $t \leftrightarrow x$ symmetry, alluded to in section 3,[7] is now obvious.

[7] Although this discrete symmetry is peculiar to all OT-G_2 solutions, not just the separable ones are considered here.

From the contracted Bianchi identity $T^{\mu\nu}_{;\nu} = 0$ we get

$$\left(\Box\Phi - \frac{\mathrm{d}V(\Phi)}{\mathrm{d}\Phi}\right)\Phi^{,\mu} = 0 \tag{50}$$

which, upon computing $\Box\Phi$, gives

$$\frac{\mathrm{d}V(\Phi)}{\mathrm{d}\Phi} = \mathrm{e}^{-2f_1}[-(\dot{f} + \tau_2 f) + (h' + \lambda_2 h)] \tag{51}$$

which is the Klein–Gordon equation governing the dynamics of the scalar field.

Turning now our attention to (49) which is the 01 field equation; and assuming $h(x) \neq 0$, we can divide through by $h(x)$ and differentiate next with respect to x thus getting

$$\tau_1(\lambda_2/h)' + \tau_2\big(\lambda_1/h - \tfrac{1}{2}\lambda_2/h\big)' - 2\tau_3(\lambda_3/h)' = 0. \tag{52}$$

Thus, three possibilities arise depending on the maximum number of linearly independent functions of t amongst τ_1, τ_2 and τ_3: three independent functions of t and just one of x, two independent functions of t and two of x, and just one linearly independent function of t and three of x; the first and last cases referred to being equivalent on account of the $t \leftrightarrow x$ discrete symmetry that these solutions possess.

The analysis of each possibility has been carried out using dynamical systems techniques, and it turns out to be very rich and interesting but rather lengthy; the corresponding results will be presented elsewhere.

6. Examples

We next present two families of solutions to the field equations obtained in the previous section.

Example 1

This is a one-parameter family of explicit solutions which has been obtained from the field equations above. Although we do not claim that it is of any particular physical significance, we would like to point out that it possesses some characteristic geometrical features: it admits two conformal Killing vectors, it is conformally related to a static metric still representing a scalar field and, for a particular value of the parameter c_1, it becomes a LRS spacetime.

Thus, consider the following values of τ_A's and λ_A's:

$$\tau_1 = \frac{1}{2}\tau_2 \qquad \tau_2 = \frac{1}{\big(c_1^2 - 1/2\big)t + k} \qquad \tau_3 = 0 \tag{53}$$

$$\lambda_1 = c_1\lambda_2 \qquad \lambda_2 = \frac{1}{x} \qquad \lambda_3 = \sqrt{c_1 - 1/4}\,\lambda_2 \tag{54}$$

where $c_1 \geqslant 1/4$. It can be easily seen that they satisfy the field equations (42)–(49) and imply for the scalar field the following expression:

$$\Phi(t,x) = \Phi_0 + \frac{c_1}{c_1^2 - 1/2}\,\ln\big[\big(c_1^2 - 1/2\big)t + k\big] + \ln(x) \tag{55}$$

whereas for the potential one has

$$V = V_0\exp(-2c_1\Phi). \tag{56}$$

The line element for this spacetime reads

$$\mathrm{d}s^2 = \big[\big(c_1^2 - 1/2\big)t + k\big]^{\frac{1}{(c_1^2-1/2)}}\{A^2x^{2c_1}[-\mathrm{d}t^2 + \mathrm{d}x^2] + B^2x[M^2C^2x^{2\sqrt{c_1-1/4}}\mathrm{d}y^2 + M^{-2}C^{-2}x^{-2\sqrt{c_1-1/4}}\mathrm{d}z^2]\}$$

where A, B, C and M are constants. Shifting now the time origin and re-scaling the coordinates, the above metric can be re-written as

$$ds^2 = A^2 t^{\frac{1}{(c_1^2-1/2t)}} x^{2c_1}\{[-dt^2 + dx^2] + x^{1-2c_1}[x^{2\sqrt{c_1-1/4}}dy^2 + x^{-2\sqrt{c_1-1/4}}dz^2]\} \tag{57}$$

where A is a scaling constant and c_1 is a parameter restricted only by $c_1 \geqslant 1/4$. This solution is included, except for the scale factor A, in a two-parameter family of solutions that appears in [6], and also in [23] (see equation (39) of that reference). In both cases the metric was found assuming a perfect fluid content for the spacetime and the existence of a conformal Killing vector (two, in fact) in the former case. As a perfect fluid it has no barotropic equation of state (we refer the reader to [6] for further details). The two conformal Killing vectors that the metric admits are $\vec{X}_1 = \partial_t$ which commutes with the two KVs generating the G_2, and $\vec{X}_2 = 2t\partial_t + 2x\partial_x + (2c_1 - 2\sqrt{c_1 - 1/4} + 1)y\partial_y + (2c_1 + 2\sqrt{c_1 - 1/4} + 1)z\partial_z$ which does not.

A few comments regarding the region where the above solutions exist are in order here: first of all, note that for $t = 0$ the metric is singular (the analysis of the structure of this singularity, though, is beyond the scope of the present work). Next, note that if $c_1^2 - 1/2$ is irrational, then $t^{1/(c_1^2-1/2)}$ is only defined for $t > 0$ and takes only positive values, and therefore, there are no problems regarding the signature of the metric. For those values of c_1 that make $c_1^2 - 1/2$ rational, put $1/\left(c_1^2 - 1/2\right) = m/n$; there are then three different cases to be distinguished:

(1) m is even and n is odd, then $t^{m/n}$ exists for both positive and negative values of t (recall $t = 0$ is a singularity) and takes only positive values (no signature problems).
(2) m is odd and n is even, $t^{m/n}$ exists only for positive values of t but the function as such is two-valued (it takes positive and negative values), in that case we just take the positive values and, again, there are no signature problems.
(3) m and n are odd, $t^{m/n}$ exists then for both $t > 0$ and $t < 0$ and takes the same sign as t; this causes a problem in the signature of the metric in the case $t < 0$; it would seem, though, that in this case, the coordinates t and x exchange their causal character thus giving rise to another spacetime (or an extension of the spacetime obtained for $t > 0$), but in order for this to be consistent in terms of signature it must be that

$$x^{2c_1} > 0 \qquad \text{and} \qquad x^{1\pm2\sqrt{c_1-1/4}} < 0 \tag{58}$$

which in turn implies that either $x > 0$ and $1 \pm 2\sqrt{c_1 - 1/4} = p_\pm/q_\pm$ with $p_\pm$ odd and $q_\pm$ even (and take then the negative branches of the two-valued functions $x^{1\pm2\sqrt{c_1-1/4}}$ which seems rather 'unnatural'); or else $x < 0$ and $1 \pm 2\sqrt{c_1 - 1/4} = r_\pm/s_\pm$ with $r_\pm$ and $s_\pm$ odd and $2c_1 = a/b$ with a even and b odd. A careful but otherwise simple analysis of these two possibilities reveals that they are incompatible with $1/\left(c_1^2 - 1/2\right) = m/n$ with both m and n odd.

Regarding the coordinate x, the metric is also singular in $x = 0$ and similar remarks to those above regarding the nature (rational or irrational) of the exponents occurring in the metric functions also apply here. Thus, if $2c_1$ (and therefore $1 \pm 2\sqrt{c_1 - 1/4}$) is irrational, the metric is only defined for positive values of x; if $1 \pm 2\sqrt{c_1 - 1/4}$ are rational (and therefore $2c_1$ is also rational), the metric can also exist for negative values of x maintaining the correct signature (see above). Thus, if $1 + 2\sqrt{c_1 - 1/4} = p/q$ with p even and q odd it follows that $1 - 2\sqrt{c_1 - 1/4} = r/s$ with r even and s odd as well, also $2c_1 = a/b$ with a,b both odd; in this case the metric exists for both positive and negative values of x but it has the correct signature only for positive values of x, note that then $1/\left(c_1^2 - 1/2\right) = m/n$ with m even and n odd and therefore the metric exists and has the correct signature for both positive and negative values of t. If p is odd and q even, then so are r and s respectively and the metric exists only for positive values of x, in this case $1/(c_1^2 - 1/2) = m/n$ with m even and n odd and therefore the metric exists and has the correct signature for both positive and negative values of t. Finally, if p and q are odd the same holds for r and s, while $2c_1 = a/b$ with a odd and b even, thus, the metric exists only for positive values of x, as for m and n it follows as before that they are even and odd, respectively, therefore the metric exists and has the correct signature for all values of t (except $t = 0$ which is singular). The behaviour of the metric obtained by making use of the $t \leftrightarrow x$ symmetry follows easily from the above remarks.

Note that for the special value $c_1 = 1/4$ the spacetime admits a further Killing vector (namely, $\vec{\zeta} = y\partial_x - x\partial_y$) which acts on the same (flat) two-dimensional orbits as $\vec{\xi} = \partial_x$ and $\vec{\eta} = \partial_y$ (i.e. it is an LRS spacetime). The line element reads in this case

$$ds^2 = A^2 t^{-16/7}[-x^{1/2}dt^2 + x^{1/2}dx^2 + x(dy^2 + dz^2)] \tag{59}$$

and although the metric functions exist for both positive and negative values of t it only has the correct signature for $t > 0$, as $x > 0$ necessarily.

Also, if one considers the following static spacetime, with a metric conformal to the above given by

$$d\bar{s}^2 = A^2[x^{2c_1}(-dt^2 + dx^2) + x(x^{2\sqrt{c_1 - 1/4}}dy^2 + x^{-2\sqrt{c_1-1/4}}dz^2)] \tag{60}$$

it is easy to see that it also corresponds to a scalar field $\Phi(x) = \ln \Phi_0 x$ with potential identically zero.

Example 2: self-similar solutions

We next present a class of solutions which are self-similar; that is: they admit a proper homothetic vector field. They all correspond to τ_As being constant and different from zero and have been obtained assuming that there is just one linearly independent function amongst λ_1, λ_2 and λ_3. Thus, put $\tau_1 = \alpha$, $\tau_2 = \beta$ and $\tau_3 = \gamma$; taking then (25) into account as well as the field equations for the λ_As subject to the above restriction, one can (after re-scaling coordinates) write the line element in this case as:

$$ds^2 = e^{2\alpha t} H^{2c_1}(-dt^2 + dx^2) + e^{2\beta t} H(e^{2\gamma t} H^{2c_3} dy^2 + e^{-2\gamma t} H^{-2c_3} dz^2) \tag{61}$$

where $H(x) = \cosh a(x - x_0)$, $\ln a(x - x_0)$, $\cos a(x - x_0)$ is the primitive of $h(x)$ depending on the sign of the constant K in (47).

This solution coincides precisely with that given in equation (37) of [23] (see also [10]) with the following identifications $H = \exp(c_2 X)$, $2\alpha = 2\psi_0 + \phi_0$, $2\beta = \phi_0$, $\gamma = \varphi_0$, and $c_1 = \bar{c}_1/c_2$, $c_3 = \bar{c}_3/c_2$, where the barred constants stand for those appearing in [23] without bar.

It is then very easy to see from the homothetic equations (3) specialized to this metric that the vector field

$$\vec{X} = \frac{k}{2\alpha}\partial_t + k\left[1 - \frac{1}{\alpha}(\beta + 2\gamma)\right] y\partial_y + k\left[1 - \frac{1}{\alpha}(\beta - 2\gamma)\right] z\partial_z \tag{62}$$

is a homothetic vector field for it with homothetic constant $k \neq 0$. This is so for any value of the constants α, β and γ; and also, this is independent of the particular value that the function $H(x)$ might take, the field equations (42)–(44) imposing only restrictions on the values of α, β and γ; thus for instance in the case $H = \cosh a(x - x_0)$ it follows that either

$$\tau_2 = \pm a\sqrt{\frac{c_1^2 + 1/2}{c_1^2 - 1/2}} \qquad \tau_3 = \pm a\sqrt{c_1 - 1/4}\sqrt{\frac{c_1^2 - 1/2}{c_1^2 + 1/2}} \qquad \tau_1 = \frac{1}{2}\left[\tau_2 + 2\frac{c_1 + 1/2}{c_3}\tau_3\right] \tag{63}$$

where $c_1 = c_3^2 + 1/4$ and the potential is $V(\Phi) = V_0 \exp(-2c_1\Phi)$; or else

$$\tau_2 = \tau_3 = \pm a \qquad \tau_1 = \pm(c_1 + 1)a \tag{64}$$

and $V(\Phi) = 0$.

7. Conclusions

We have studied scalar field spacetimes, first at a very general level and then as sources of inhomogeneous G_2 cosmological models.

At the general level, we have discussed that the manifold of one such spacetime can be decomposed as the disjoint union of three sets, two open ones on which the energy–momentum tensor has (diagonal) constant Segre type on each of them, and a closed set (with no interior whenever the energy–momentum is analytic) which is a kind of border between the former two. The dominant energy condition has proved to be satisfied all over the manifold provided the potential is positive. Also, this decomposition has been studied in connection with inflation (suggesting a criterion to check whether a given model inflates or not in the case of inhomogeneous spacetimes with a scalar field content), and with the strong energy condition as well, showing that its violation is generical in one of the open components above.

Further, and still at this general level, we have shown a few straightforward results concerning isometries and homotheties in scalar field spacetimes; in particular it has been proved that if one such spacetime admits a homothety, the potential must be necessarily of the exponential type (or zero). Note that this form of the potential was assumed as a further restriction in the analysis of the self-similar family carried out in references [3, 7], whereas our result implies that this is not, in fact, a restriction but rather a consequence of the geometry assumed.

We have next turned our attention to the inhomogeneous G_2 orthogonally transitive models, giving a result that characterizes scalar field spacetimes in this geometric setup; the result is not completely original in the sense that a similar one for perfect fluids already exists in the literature (see [6]).

We have then concentrated on those diagonal G_2 orthogonally transitive models which are separable in coordinates adapted to the Killing vectors and have shown that this necessarily implies separability in the scalar field; which—while reasonable and expectable—does not follow immediately. Note that this is an extra assumption, made for the sake of simplicity, in the literature (see [7, 8, 16, 17] and the references cited therein). We show though that again this is a consequence of previous assumptions on the form of the metric.

Following this, we have shown that the field equations separate completely into two subsystems of equations, one depending on the time variable t and the other on one of the spacelike coordinates x, both of them can be written as first order[8] autonomous systems of differential equations. Three cases appear depending on the number of linearly independent functions; three independent functions of t and just one of x, two independent functions of t, and just one linearly independent function of t and three of x; the first and last case referred to being equivalent on account of the $t \leftrightarrow x$ discrete symmetry that these solutions possess. Again, all this is completely general and does not depend on any further simplifying assumption. The families of solutions considered in the aforementioned references, can be seen to be special instances of one of the above cases. Finally, two families of simple but illustrative examples are given.

[8] After suitably redefining the first order derivative of the metric functions as the new unknown functions.

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